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Stochastic Differential Equations
Homework Sheet 1 - solutions

Problem 1. Let P be a probability measure. Suppose $P(A) = \frac{3}{4}$ and $P(B) = \frac{1}{3}$. Show that

$$\frac{1}{12} \leq P(A \cap B) \leq \frac{1}{3}.$$

Solution. For any events A, B,

$$P(A \cap B) \leq \min\{P(A), P(B)\}, \qquad P(A \cap B) \geq P(A) + P(B) - 1. \tag{1}$$

The first inequality follows from the fact that $A \cap B$ is a subset of A and of B, and from the monotonicity of P. The second inequality follows from De Morgan's law and the union bound:

$$P(A \cap B) = 1 - P(A^c \cup B^c) \geq 1 - P(A^c) - P(B^c) = P(A) + P(B) - 1.$$

Hence, by (1),

$$P(A \cap B) \leq \min\left(\tfrac{3}{4}, \tfrac{1}{3}\right) = \tfrac{1}{3}, \qquad P(A \cap B) \geq \tfrac{3}{4} + \tfrac{1}{3} - 1 = \tfrac{1}{12}.$$

Therefore,

$$\frac{1}{12} \leq P(A \cap B) \leq \frac{1}{3}.$$

Problem 2. Let Ω be an infinite set (countable or not) and let

$$\mathcal{G} = \{A \subseteq \Omega : \ A \text{ is finite}\} \cup \{A \subseteq \Omega : \ \Omega \setminus A \text{ is finite}\}$$

be the family of all finite or cofinite subsets of Ω. Show that $\mathcal{G}$ is an algebra of sets but not a σ-algebra.

Solution. *Algebra.* (i) $\varnothing \in \mathcal{G}$ (finite) and $\Omega \in \mathcal{G}$ (cofinite). (ii) If $A \in \mathcal{G}$, then A is finite or cofinite. In either case A^c is cofinite or finite, respectively; hence $A^c \in \mathcal{G}$. (iii) If $A, B \in \mathcal{G}$, then:

- finite $\cup$ finite is finite;
- finite $\cup$ cofinite is cofinite, because (finite $\cup$ cofinite)^c=finite$^c \cap$ cofinite^c is finite;
- cofinite $\cup$ cofinite is cofinite because $(A \cup B)^c = A^c \cap B^c$, an intersection of finite sets, hence finite.

Thus $\mathcal{G}$ is closed under complements and finite unions, so it is an algebra.

Not a σ-algebra. We produce a countable family $(A_n)_{n\geq 1} \subset \mathcal{G}$ whose union is not in $\mathcal{G}$.

Case 1: Ω countably infinite. Identify Ω with $\mathbb{N}$ and set $A_n = \{2n\}$ (singletons, hence in $\mathcal{G}$). Then

$$\bigcup_{n=1}^{\infty} A_n = \{2, 4, 6, \dots\}$$

is infinite and has infinite complement $\{1, 3, 5, \dots\}$, so it is neither finite nor cofinite. Hence it is not in $\mathcal{G}$.

Case 2: Ω uncountable. Choose distinct points $x_1, x_2, \cdots \in \Omega$ and let $A_n = \{x_n\}$. Each A_n is finite, so $A_n \in \mathcal{G}$. But

$$S := \bigcup_{n=1}^{\infty} A_n = \{x_n : \ n \geq 1\}$$

is countably infinite, and since Ω is uncountable, $\Omega \setminus S$ is infinite; thus S is neither finite nor cofinite, so $S \notin \mathcal{G}$.

In either case, $\mathcal{G}$ fails to be closed under countable unions; therefore it is not a σ-algebra.

Problem 3. Let $\{H_1, \dots, H_k\}$ be a finite partition of Ω with $P(H_i) > 0$ for all i.

(a) Prove the *law of total probability*:

$$P(B) = \sum_{i=1}^{k} P(B \mid H_i)\, P(H_i).$$

(b) Use this result to derive the following formula

$$P(H_j \mid B) = \frac{P(B \mid H_j)\, P(H_j)}{\sum_{i=1}^{k} P(B \mid H_i)\, P(H_i)}.$$

under the assumption $P(B) \neq 0$.

Solution. (a) Law of total probability. Since $\{H_1, \dots, H_k\}$ is a partition of Ω, the sets $B \cap H_i$ are pairwise disjoint and

$$B = \bigcup_{i=1}^{k} (B \cap H_i).$$

By finite additivity,

$$P(B) = \sum_{i=1}^{k} P(B \cap H_i) = \sum_{i=1}^{k} P(B \mid H_i)\, P(H_i),$$

where $P(H_i) > 0$ ensures $P(B \mid H_i)$ is well-defined.

(b) Bayes rule with a partition. Fix $j \in \{1, \ldots, k\}$ with $P(H_j) > 0$. By the definition of conditional probability,

$$P(H_j \mid B) = \frac{P(H_j \cap B)}{P(B)} = \frac{P(B \mid H_j)\, P(H_j)}{P(B)}.$$

Substituting the result from part (a) for $P(B)$ gives

$$P(H_j \mid B) = \frac{P(B \mid H_j)\, P(H_j)}{\sum_{i=1}^{k} P(B \mid H_i)\, P(H_i)}.$$

Problem 4. A disease affects 1% of a population. A test for the disease has:

- sensitivity $P(+ \mid D) = 0.99$ (true positive rate). That is, a person who has the disease will have a positive test result with probability 0.99.
- specificity $P(- \mid \overline{D}) = 0.95$ (true negative rate). That is, a person who does not have the disease will have a negative test result with probability 0.95.

If a randomly chosen person tests positive, what is $P(D \mid +)$, i.e., the probability that a positively tested person has the disease? Hint: Use Problem 3.

Solution. Let $\Omega = \{\text{all people in the population}\}$. We are interested in subsets:

- D - people having the disease.
- $\overline{D}$ - people not having the disease.
- $+$ - people positively tested.
- $-$ - people negatively tested.

Obviously $\Omega = D \cup \bar{D}$ so we can choose the partition $H_1 = D$, $H_2 = \bar{D}$. We also have $(+) \cup (-) = \Omega$.

We have some information about the probability P on Ω: $P(D) = 0.01$, hence $P(\overline{D}) = 0.99$. From the specificity we have $P(+ \mid \overline{D}) = 1 - 0.95 = 0.05$. Here we used that

$$P(+|\bar{D}) + P(-|\bar{D}) = \frac{P(+ \cap \bar{D})}{P(\bar{D})} + \frac{P(- \cap \bar{D})}{P(\bar{D})} = \frac{P(\bar{D})}{P(\bar{D})} = 1. \tag{2}$$

By Bayes' rule,

$$P(D \mid +) = \frac{P(+ \mid D)P(D)}{P(+ \mid D)P(D) + P(+ \mid \overline{D})P(\overline{D})} = \frac{0.99 \cdot 0.01}{0.99 \cdot 0.01 + 0.05 \cdot 0.99} = \frac{1}{6} \approx 16.7\%.$$

Therefore, $P(D \mid +) = \dfrac{1}{6} \approx 16.7\%$.

So sensitivity and specificity above 90% does not mean that the test is good. The reason is that with such small proportion of population having the disease, specificity 0.95 will generate plenty of false positive results.

Problem 5. A box contains two coins:

- Coin A is fair: $P(H \mid A) = \frac{1}{2}$,
- Coin B is biased: $P(H \mid B) = \frac{3}{4}$.

One coin is chosen at random (so $P(A) = P(B) = \frac{1}{2}$) and tossed.

(a) The first toss is heads. Compute $P(B \mid H)$.

(b) The coin is tossed again and the second toss is also heads. Compute $P(B \mid HH)$.

(c) (Generalization) After n heads in a row, express $P(B \mid H^n)$ in closed form.

Solution. (a) By Bayes' rule,

$$P(B \mid H) = \frac{P(H \mid B)P(B)}{P(H \mid B)P(B) + P(H \mid A)P(A)} = \frac{\frac{3}{4} \cdot \frac{1}{2}}{\frac{3}{4} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}} = \frac{3}{5} = 0.6.$$

(b) Conditioned on the chosen coin, tosses are independent, so

$$P(HH \mid B) = \left(\tfrac{3}{4}\right)^2 = \tfrac{9}{16}, \qquad P(HH \mid A) = \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{4}.$$

Thus,

$$P(B \mid HH) = \frac{P(HH \mid B)P(B)}{P(HH \mid B)P(B) + P(HH \mid A)P(A)} = \frac{\frac{9}{16} \cdot \frac{1}{2}}{\frac{9}{16} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2}} = \frac{\frac{9}{32}}{\frac{9}{32} + \frac{1}{8}} = \frac{9}{13} \approx 0.6923.$$

(c) After n heads in a row,

$$P(H^n \mid B) = \left(\tfrac{3}{4}\right)^n, \qquad P(H^n \mid A) = \left(\tfrac{1}{2}\right)^n.$$

Hence,

$$P(B \mid H^n) = \frac{P(H^n \mid B)P(B)}{P(H^n \mid B)P(B) + P(H^n \mid A)P(A)} = \frac{(\frac{3}{4})^n \cdot \frac{1}{2}}{(\frac{3}{4})^n \cdot \frac{1}{2} + (\frac{1}{2})^n \cdot \frac{1}{2}} = \frac{3^n}{3^n + 2^n}.$$

Problem 6. Let $(\Omega, \mathcal{F}, P)$ be a probability space.

(a) (**Continuity from below**) Let $(A_n)_{n \geq 1}$ be an increasing sequence of events ($A_n \subseteq A_{n+1}$ for all n). Show that

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} P(A_n).$$

(b) (**Continuity from above**) Let $(B_n)_{n\geq 1}$ be a decreasing sequence of events ($B_{n+1} \subseteq B_n$ for all n). Show that

$$P\left(\bigcap_{n=1}^{\infty} B_n\right) \;=\; \lim_{n\to\infty} P(B_n).$$

Solution. (a) Define $C_n := A_n \setminus A_{n-1}$ (with $A_0 = \varnothing$). Then $C_n \cap C_m = \varnothing$ for $n \neq m$ and $\bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} A_n$. Moreover $A_n = \bigcup_{k=1}^{n} C_k$, hence

$$P(A_n) = \sum_{k=1}^{n} P(C_k).$$

Taking $n \to \infty$ gives, by σ-additivity,

$$\lim_{n\to\infty} P(A_n) = \sum_{k=1}^{\infty} P(C_k) = P\left(\bigcup_{n=1}^{\infty} C_n\right) = P\left(\bigcup_{n=1}^{\infty} A_n\right).$$

(b) Let $D_n := B_1 \setminus B_n$. Then (D_n) is increasing, and

$$\bigcup_{n=1}^{\infty} D_n = B_1 \setminus \bigcap_{n=1}^{\infty} B_n.$$

By part (a),

$$\lim_{n\to\infty} P(D_n) = P\left(\bigcup_{n=1}^{\infty} D_n\right) = P(B_1) - P\left(\bigcap_{n=1}^{\infty} B_n\right).$$

But $P(D_n) = P(B_1) - P(B_n)$, hence

$$\lim_{n\to\infty} P(B_n) = P\left(\bigcap_{n=1}^{\infty} B_n\right).$$

Problem 7. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(A_n)_{n\geq 1}$ a sequence of events. Recall the *limsup* ("infinitely often") event

$$\limsup_{n\to\infty} A_n \;=\; \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \;=\; \{\omega :\ \omega \in A_n \text{ for infinitely many } n\}.$$

Prove:

(i) If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then

$$P\left(\limsup_{n\to\infty} A_n\right) = 0.$$

(ii) If the events (A_n) are independent and $\sum_{n=1}^{\infty} P(A_n) = \infty$, then

$$P\left(\limsup_{n\to\infty} A_n\right) = 1.$$

Hint for (ii): Use independence to write $P\big(\bigcap_{k=n}^{m} A_k^c\big) = \prod_{k=n}^{m}(1-P(A_k))$ and the inequality $1-x \le e^{-x}$ for $x \in [0,1]$.

Solution. (i) First Borel–Cantelli. Set $B_n := \bigcup_{k=n}^{\infty} A_k$. Then $(B_n)_{n\in\mathbb{N}}$ is a decreasing sequence of events and $\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} B_n$. By subadditivity,

$$P(B_n) \;\le\; \sum_{k=n}^{\infty} P(A_k).$$

Since $\sum_{k=1}^{\infty} P(A_k) < \infty$, the tail sums $\sum_{k=n}^{\infty} P(A_k) \to 0$ as $n \to \infty$. By Problem 6(b) (continuity of probability from above), applied to the decreasing sequence (B_n),

$$P\left(\limsup_{n\to\infty} A_n\right) = P\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} P(B_n) \le \lim_{n\to\infty} \sum_{k=n}^{\infty} P(A_k) = 0.$$

(ii) Second Borel–Cantelli. Let $C_n := \bigcap_{k=n}^{\infty} A_k^c$ be the event that no A_k occurs for $k \ge n$. Then

$$\left(\limsup_{n\to\infty} A_n\right)^c = \bigcup_{n=1}^{\infty} C_n, \qquad \text{hence} \qquad P\left(\limsup_{n\to\infty} A_n\right) = 1 - \lim_{n\to\infty} P(C_n),$$

using that (C_n) is increasing as less sets get intersected when n increases.

Since independence of (A_k) implies independence of (A_k^c), for each $m > n$,

$$P\left(\bigcap_{k=n}^{m} A_k^c\right) = \prod_{k=n}^{m}(1-P(A_k)) \;\le\; \exp\left(-\sum_{k=n}^{m} P(A_k)\right),$$

since $1-x \le e^{-x}$ for $x \in [0,1]$. Now we write

$$P(C_n) \;=\; \lim_{m\to\infty} P\left(\bigcap_{k=n}^{m} A_k^c\right) \;\le\; \exp\left(-\sum_{k=n}^{\infty} P(A_k)\right),$$

where, in the first step, we used the previous exercise and

$$C_n = \bigcap_{k=n}^{\infty} A_k^c = \bigcap_{m=n}^{\infty} \bigcap_{k=n}^{m} A_k^c = \bigcap_{m=n}^{\infty} D_{n,m}, \tag{3}$$

for a decreasing (in m) family $D_{n,m}$.

Since $\sum_{k=1}^{\infty} P(A_k) = \infty$, removing finitely many terms leaves $\sum_{k=n}^{\infty} P(A_k) = \infty$ for every n. The preceding bound therefore gives $P(C_n) = 0$ for every n. Hence,

$$P\left(\limsup_{n\to\infty} A_n\right) = 1 - \lim_{n\to\infty} P(C_n) = 1.$$

To be discussed in class: 09.10.2026