

Stochastic Differential Equations

Homework Sheet 1 - solutions

Problem 1. Let P be a probability measure. Suppose $P(A) = \frac{3}{4}$ and $P(B) = \frac{1}{3}$. Show that

$$\frac{1}{12} \leq P(A \cap B) \leq \frac{1}{3}.$$

Problem 2. Let Ω be an infinite set (countable or not) and let

$$\mathcal{G} = \{A \subseteq \Omega : A \text{ is finite}\} \cup \{A \subseteq \Omega : \Omega \setminus A \text{ is finite}\}$$

be the family of all finite or cofinite subsets of Ω . Show that $\mathcal{G}$ is an algebra of sets but not a σ -algebra.

These problems may be still discussed on 16.10.2026. Solutions will be posted afterwards.

Problem 3. Let $\{H_1, \dots, H_k\}$ be a finite partition of Ω with $P(H_i) > 0$ for all i .

(a) Prove the *law of total probability*:

$$P(B) = \sum_{i=1}^k P(B \mid H_i) P(H_i).$$

(b) Use this result to derive the following formula

$$P(H_j \mid B) = \frac{P(B \mid H_j) P(H_j)}{\sum_{i=1}^k P(B \mid H_i) P(H_i)}.$$

under the assumption $P(B) \neq 0$.

Solution. (a) Law of total probability. Since $\{H_1, \dots, H_k\}$ is a partition of Ω , the sets $B \cap H_i$ are pairwise disjoint and

$$B = \bigcup_{i=1}^k (B \cap H_i).$$

By finite additivity,

$$P(B) = \sum_{i=1}^k P(B \cap H_i) = \sum_{i=1}^k P(B \mid H_i) P(H_i),$$

where $P(H_i) > 0$ ensures $P(B | H_i)$ is well-defined.

(b) Bayes rule with a partition. Fix $j \in \{1, \dots, k\}$ with $P(H_j) > 0$. By the definition of conditional probability,

$$P(H_j | B) = \frac{P(H_j \cap B)}{P(B)} = \frac{P(B | H_j) P(H_j)}{P(B)}.$$

Substituting the result from part (a) for $P(B)$ gives

$$P(H_j | B) = \frac{P(B | H_j) P(H_j)}{\sum_{i=1}^k P(B | H_i) P(H_i)}.$$

Problem 4. A disease affects 1% of a population. A test for the disease has:

- sensitivity $P(+ | D) = 0.99$ (true positive rate). That is, a person who has the disease will have a positive test result with probability 0.99.
- specificity $P(- | \bar{D}) = 0.95$ (true negative rate). That is, a person who does not have the disease will have a negative test result with probability 0.95.

If a randomly chosen person tests positive, what is $P(D | +)$, i.e., the probability that a positively tested person has the disease? Hint: Use Problem 3.

Solution. Let $\Omega = \{\text{all people in the population}\}$. We are interested in subsets:

- D - people having the disease.
- $\bar{D}$ - people not having the disease.
- $+$ - people positively tested.
- $-$ - people negatively tested.

Obviously $\Omega = D \cup \bar{D}$ so we can choose the partition $H_1 = D$, $H_2 = \bar{D}$. We also have $(+) \cup (-) = \Omega$.

We have some information about the probability P on Ω : $P(D) = 0.01$, hence $P(\bar{D}) = 0.99$. From the specificity we have $P(+ | \bar{D}) = 1 - 0.95 = 0.05$. Here we used that

$$P(+ | \bar{D}) + P(- | \bar{D}) = \frac{P(+ \cap \bar{D})}{P(\bar{D})} + \frac{P(- \cap \bar{D})}{P(\bar{D})} = \frac{P(\bar{D})}{P(\bar{D})} = 1. \quad (1)$$

By Bayes' rule,

$$P(D | +) = \frac{P(+ | D)P(D)}{P(+ | D)P(D) + P(+ | \bar{D})P(\bar{D})} = \frac{0.99 \cdot 0.01}{0.99 \cdot 0.01 + 0.05 \cdot 0.99} = \frac{1}{6} \approx 16.7\%.$$

Therefore, $P(D | +) = \frac{1}{6} \approx 16.7\%$.

So sensitivity and specificity above 90% does not mean that the test is good. The reason is that with such small proportion of population having the disease, specificity 0.95 will generate plenty of false positive results.

Problem 5. A box contains two coins:

- Coin A is fair: $P(H | A) = \frac{1}{2}$,
- Coin B is biased: $P(H | B) = \frac{3}{4}$.

One coin is chosen at random (so $P(A) = P(B) = \frac{1}{2}$) and tossed.

- The first toss is heads. Compute $P(B | H)$.
- The coin is tossed again and the second toss is also heads. Compute $P(B | HH)$.
- (Generalization) After n heads in a row, express $P(B | H^n)$ in closed form.

Solution. (a) By Bayes' rule,

$$P(B | H) = \frac{P(H | B)P(B)}{P(H | B)P(B) + P(H | A)P(A)} = \frac{\frac{3}{4} \cdot \frac{1}{2}}{\frac{3}{4} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}} = \frac{3}{5} = 0.6.$$

(b) Conditioned on the chosen coin, tosses are independent, so

$$P(HH | B) = \left(\frac{3}{4}\right)^2 = \frac{9}{16}, \quad P(HH | A) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Thus,

$$P(B | HH) = \frac{P(HH | B)P(B)}{P(HH | B)P(B) + P(HH | A)P(A)} = \frac{\frac{9}{16} \cdot \frac{1}{2}}{\frac{9}{16} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2}} = \frac{\frac{9}{32}}{\frac{9}{32} + \frac{1}{8}} = \frac{9}{13} \approx 0.6923.$$

(c) After n heads in a row,

$$P(H^n | B) = \left(\frac{3}{4}\right)^n, \quad P(H^n | A) = \left(\frac{1}{2}\right)^n.$$

Hence,

$$P(B | H^n) = \frac{P(H^n | B)P(B)}{P(H^n | B)P(B) + P(H^n | A)P(A)} = \frac{\left(\frac{3}{4}\right)^n \cdot \frac{1}{2}}{\left(\frac{3}{4}\right)^n \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^n \cdot \frac{1}{2}} = \frac{3^n}{3^n + 2^n}.$$

Problem 6. Let $(\Omega, \mathcal{F}, P)$ be a probability space.

- (**Continuity from below**) Let $(A_n)_{n \geq 1}$ be an increasing sequence of events ($A_n \subseteq A_{n+1}$ for all n). Show that

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} P(A_n).$$

- (b) **(Continuity from above)** Let $(B_n)_{n \geq 1}$ be a decreasing sequence of events ($B_{n+1} \subseteq B_n$ for all n). Show that

$$P\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n \rightarrow \infty} P(B_n).$$

Solution. (a) Define $C_n := A_n \setminus A_{n-1}$ (with $A_0 = \emptyset$). Then $C_n \cap C_m = \emptyset$ for $n \neq m$ and $\bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} A_n$. Moreover $A_n = \bigcup_{k=1}^n C_k$, hence

$$P(A_n) = \sum_{k=1}^n P(C_k).$$

Taking $n \rightarrow \infty$ gives, by σ -additivity,

$$\lim_{n \rightarrow \infty} P(A_n) = \sum_{k=1}^{\infty} P(C_k) = P\left(\bigcup_{n=1}^{\infty} C_n\right) = P\left(\bigcup_{n=1}^{\infty} A_n\right).$$

- (b) Let $D_n := B_1 \setminus B_n$. Then (D_n) is increasing, and

$$\bigcup_{n=1}^{\infty} D_n = B_1 \setminus \bigcap_{n=1}^{\infty} B_n.$$

By part (a),

$$\lim_{n \rightarrow \infty} P(D_n) = P\left(\bigcup_{n=1}^{\infty} D_n\right) = P(B_1) - P\left(\bigcap_{n=1}^{\infty} B_n\right).$$

But $P(D_n) = P(B_1) - P(B_n)$, hence

$$\lim_{n \rightarrow \infty} P(B_n) = P\left(\bigcap_{n=1}^{\infty} B_n\right).$$

Problem 7. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(A_n)_{n \geq 1}$ a sequence of events. Recall the *limsup* (“infinitely often”) event

$$\limsup_{n \rightarrow \infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \{\omega : \omega \in A_n \text{ for infinitely many } n\}.$$

Prove:

- (i) If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then

$$P\left(\limsup_{n \rightarrow \infty} A_n\right) = 0.$$

- (ii) If the events (A_n) are independent and $\sum_{n=1}^{\infty} P(A_n) = \infty$, then

$$P\left(\limsup_{n \rightarrow \infty} A_n\right) = 1.$$

Hint for (ii): Use independence to write $P(\bigcap_{k=n}^m A_k^c) = \prod_{k=n}^m (1 - P(A_k))$ and the inequality $1 - x \leq e^{-x}$ for $x \in [0, 1]$.

Solution. (i) First Borel–Cantelli. Set $B_n := \bigcup_{k=n}^{\infty} A_k$. Then $(B_n)_{n \in \mathbb{N}}$ is a decreasing sequence of events and $\limsup_{n \rightarrow \infty} A_n = \bigcap_{n=1}^{\infty} B_n$. By subadditivity,

$$P(B_n) \leq \sum_{k=n}^{\infty} P(A_k).$$

Since $\sum_{k=1}^{\infty} P(A_k) < \infty$, the tail sums $\sum_{k=n}^{\infty} P(A_k) \rightarrow 0$ as $n \rightarrow \infty$. By Problem 6(b) (continuity of probability from above), applied to the decreasing sequence (B_n) ,

$$P\left(\limsup_{n \rightarrow \infty} A_n\right) = P\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n \rightarrow \infty} P(B_n) \leq \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} P(A_k) = 0.$$

(ii) Second Borel–Cantelli. Let $C_n := \bigcap_{k=n}^{\infty} A_k^c$ be the event that no A_k occurs for $k \geq n$. Then

$$\left(\limsup_{n \rightarrow \infty} A_n\right)^c = \bigcup_{n=1}^{\infty} C_n, \quad \text{hence} \quad P\left(\limsup_{n \rightarrow \infty} A_n\right) = 1 - \lim_{n \rightarrow \infty} P(C_n),$$

using that (C_n) is increasing as less sets get intersected when n increases.

Since independence of (A_k) implies independence of (A_k^c) , for each $m > n$,

$$P\left(\bigcap_{k=n}^m A_k^c\right) = \prod_{k=n}^m (1 - P(A_k)) \leq \exp\left(-\sum_{k=n}^m P(A_k)\right),$$

since $1 - x \leq e^{-x}$ for $x \in [0, 1]$. Now we write

$$P(C_n) = \lim_{m \rightarrow \infty} P\left(\bigcap_{k=n}^m A_k^c\right) \leq \exp\left(-\sum_{k=n}^{\infty} P(A_k)\right),$$

where, in the first step, we used the previous exercise and

$$C_n = \bigcap_{k=n}^{\infty} A_k^c = \bigcap_{m=n}^{\infty} \bigcap_{k=n}^m A_k^c = \bigcap_{m=n}^{\infty} D_{n,m}, \quad (2)$$

for a decreasing (in m) family $D_{n,m}$.

Since $\sum_{k=1}^{\infty} P(A_k) = \infty$, removing finitely many terms leaves $\sum_{k=n}^{\infty} P(A_k) = \infty$ for every n . The preceding bound therefore gives $P(C_n) = 0$ for every n . Hence,

$$P\left(\limsup_{n \rightarrow \infty} A_n\right) = 1 - \lim_{n \rightarrow \infty} P(C_n) = 1.$$

To be discussed in class: 09.10.2026