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Stochastic Differential Equations
Homework Sheet 2 - solutions

Problem 1. Let $(\Omega, \mathcal{F}, P) = (\mathbb{R}_+, \mathcal{B}, \mu)$, where $\mathbb{R}_+ = [0, \infty)$, $\mathcal{B}$ denotes the Borel σ-algebra on $[0, \infty)$ and μ is a probability measure on $[0, \infty)$ with no mass on single points, i.e. $\mu(\{x\}) = 0$ for every $x \in \mathbb{R}_+$. Define

$$X_t(\omega) = \begin{cases} 1 & \text{if } t = \omega, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

and

$$Y_t(\omega) = 0 \quad \text{for all} \quad (t, \omega) \in \mathbb{R}_+ \times \mathbb{R}_+. \tag{2}$$

Prove that $\{X_t\}_{t\in\mathbb{R}_+}$ and $\{Y_t\}_{t\in\mathbb{R}_+}$ have the same finite-dimensional distributions and that $\{X_t\}_{t\in\mathbb{R}_+}$ is a version of $\{Y_t\}_{t\in\mathbb{R}_+}$. (Note that, nevertheless, $t \mapsto Y_t(\omega)$ is continuous for all ω and $t \mapsto X_t(\omega)$ is discontinuous for all ω).

Solution. (i) Claim: Equality in law of the finite-dimensional distributions. Proof: Fix $k \in \mathbb{N}$ and times $t_1, \ldots, t_k \in \mathbb{R}_+$. Put

$$A_i := \{\omega \in \Omega : \omega = t_i\} = \{t_i\}, \qquad A := \bigcup_{i=1}^{k} A_i.$$

Since μ is non-atomic, $\mu(A_i) = 0$ for each i, hence $\mu(A) = 0$. For $\omega \notin A$ we have $X_{t_i}(\omega) = 0 = Y_{t_i}(\omega)$. Therefore, for any Borel sets $F_1, \ldots F_k \subset \mathbb{R}_+$,

$$\begin{aligned} \mu\big(\{\omega : X_{t_1}(\omega) \in F_1, \ldots, X_{t_k}(\omega) \in F_k\}\big) &= \mu\big(\{\omega : X_{t_1}(\omega) \in F_1, \ldots, X_{t_k}(\omega) \in F_k\} \backslash A\big) \\ &= \mu\big(\{\omega : Y_{t_1}(\omega) \in F_1, \ldots, Y_{t_k}(\omega) \in F_k\} \backslash A\big) \\ &= \mu\big(\{\omega : Y_{t_1}(\omega) \in F_1, \ldots, Y_{t_k}(\omega) \in F_k\}\big) \end{aligned} \tag{3}$$

(ii) Claim: X is a version of Y. Proof: For each fixed $t \geq 0$,

$$\mu\big(X_t \neq Y_t\big) = \mu\big(X_t \neq 0\big) = \mu(\{\omega = t\}) = \mu(\{t\}) = 0,$$

since μ has no mass on single points.

Problem 2. Let X and Y be two independent random variables with finite variances. Show that

$$\operatorname{var}(X + Y) = \operatorname{var}(X) + \operatorname{var}(Y).$$

Solution. Recall that for any random variable Z,

$$\operatorname{var}(Z) = E[Z^2] - (E[Z])^2.$$

We compute:

$$\operatorname{var}(X+Y) = E[(X+Y)^2] - (E[X+Y])^2.$$

Expanding the square gives

$$E[(X+Y)^2] = E[X^2] + 2E[XY] + E[Y^2],$$

and

$$E[X+Y] = E[X] + E[Y].$$

Hence

$$(E[X+Y])^2 = (E[X] + E[Y])^2 = (E[X])^2 + 2E[X]E[Y] + (E[Y])^2.$$

Subtracting, we obtain

$$\operatorname{var}(X+Y) = (E[X^2] - (E[X])^2) + (E[Y^2] - (E[Y])^2) + 2(E[XY] - E[X]E[Y]).$$

Let us call the last term $2\operatorname{Cov}(X,Y)$. Since X and Y are independent, $E[XY] = E[X]E[Y]$, and therefore

$$\operatorname{Cov}(X,Y) = 0.$$

It follows that

$$\operatorname{var}(X+Y) = \operatorname{var}(X) + \operatorname{var}(Y).$$

Remark: The result immediately generalizes to finite sums of independent random variables.

Problem 3. Let X be a real-valued random variable with finite second moment, i.e., $E[X^2] < \infty$.

(a) Let $Y \geq 0$ be any nonnegative random variable and let $a > 0$. Show that

$$P(Y \geq a) \leq \frac{E[Y]}{a}.$$

(b) Apply part (a) with $Y = (X - E[X])^2$ and $a = \varepsilon^2$ for $\varepsilon > 0$. Conclude that

$$P\big(|X - E[X]| \geq \varepsilon\big) \leq \frac{E[(X - E[X])^2]}{\varepsilon^2}.$$

(c) Recall that $\operatorname{var}(X) := E[(X - E[X])^2]$. Rewrite the bound from part (b) in terms of $\operatorname{var}(X)$.

Solution. (a) Since $Y \geq 0$ and $a > 0$,

$$E[Y] \;=\; E\big[Y\,\chi_{\{Y<a\}}\big] + E\big[Y\,\chi_{\{Y\geq a\}}\big] \;\geq\; E\big[a\,\chi_{\{Y\geq a\}}\big] \;=\; a\,P(Y \geq a).$$

Dividing by a yields $P(Y \geq a) \leq E[Y]/a$. This bound is called the Markov inequality.

(b) Take $Y = (X - E[X])^2 \geq 0$ and $a = \varepsilon^2$ with $\varepsilon > 0$. Then

$$P\big(|X - E[X]| \geq \varepsilon\big) = P\big((X - E[X])^2 \geq \varepsilon^2\big) \leq \frac{E[(X - E[X])^2]}{\varepsilon^2}.$$

(c) By definition $\mathrm{var}(X) = E[(X - E[X])^2]$, so

$$P\big(|X - E[X]| \geq \varepsilon\big) \leq \frac{\mathrm{var}(X)}{\varepsilon^2}.$$

The last two bounds are called Chebyshev's inequalities.

Problem 4. Let $\{B_t\}_{t\in\mathbb{R}_+}$ be 2-dimensional Brownian motion and set

$$D_\rho := \{\, y \in \mathbb{R}^2 \,:\, |y| < \rho\} \quad \text{for} \quad \rho > 0. \tag{4}$$

Compute $P^{x=0}(B_t \in D_\rho)$ for some fixed $t > 0$.

Solution. By the construction of the Brownian motion via the Kolmogorov's extension theorem, we have

$$P(B_t \in D_\rho) = \int_{D_\rho} p(t, 0, y) dy, \tag{5}$$

where

$$p(t, 0, y) = \frac{1}{2\pi t}\, \mathrm{e}^{-\frac{|y|^2}{2t}}, \qquad y \in \mathbb{R}^2.$$

We compute

$$P^{x=0}(B_t \in D_\rho) = \int_{|y|<\rho} p(t, 0, y)\, dy = \int_0^{2\pi}\int_0^{\rho} \frac{1}{2\pi t}\, \mathrm{e}^{-\frac{r^2}{2t}}\, r\, dr\, d\theta = \int_0^{\rho} \frac{1}{t}\, \mathrm{e}^{-\frac{r^2}{2t}}\, r\, dr.$$

Let $u = \frac{r^2}{2t}$ so that $r\, dr = t\, du$. Then

$$P^{x=0}(B_t \in D_\rho) = \int_0^{\rho^2/(2t)} \mathrm{e}^{-u}\, du = 1 - \mathrm{e}^{-\frac{\rho^2}{2t}}.$$

As expected, for small t the probability is almost 1, whereas for large t it is almost zero.

Problem 5. Show that:

(a) $I := \int_{\mathbb{R}} \mathrm{e}^{-\frac{x^2}{2\sigma^2}} dx = \sqrt{2\pi\sigma^2}$.
Hint: For example, write $I^2 = \int_{\mathbb{R}^2} \mathrm{e}^{-\frac{1}{2\sigma^2}(x^2+y^2)} dxdy$ and go to polar coordinates.

(b) $\int_{\mathbb{R}} \mathrm{e}^{ikx} \mathrm{e}^{-\frac{x^2}{2\sigma^2}} dx = \sqrt{2\pi\sigma^2}\, \mathrm{e}^{-\frac{k^2\sigma^2}{2}}$.

(c) $\int_{\mathbb{R}^d} \mathrm{e}^{i\langle k,x\rangle} \mathrm{e}^{-\frac{\langle x, C^{-1}x\rangle}{2}} dx = \sqrt{(2\pi)^d \det(C)}\, \mathrm{e}^{-\frac{\langle k, Ck\rangle}{2}}$, where C is a $d \times d$ symmetric matrix with strictly positive eigenvalues. Hint: Diagonalize the matrix, change variables and use the previous item.

Solution. (a) Let

$$I := \int_{\mathbb{R}} \mathrm{e}^{-\frac{x^2}{2\sigma^2}}\, dx.$$

Then

$$I^2 = \left(\int_{\mathbb{R}} \mathrm{e}^{-\frac{x^2}{2\sigma^2}}\, dx\right)\left(\int_{\mathbb{R}} \mathrm{e}^{-\frac{y^2}{2\sigma^2}}\, dy\right) = \int_{\mathbb{R}^2} \mathrm{e}^{-\frac{x^2+y^2}{2\sigma^2}}\, dx\, dy.$$

Passing to polar coordinates (r,θ) with $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$, we obtain

$$I^2 = \int_0^{2\pi}\int_0^{\infty} \mathrm{e}^{-\frac{r^2}{2\sigma^2}}\, r\, dr\, d\theta = 2\pi \int_0^{\infty} \mathrm{e}^{-\frac{r^2}{2\sigma^2}}\, r\, dr.$$

Let $u = \frac{r^2}{2\sigma^2}$, so $du = \frac{r}{\sigma^2}\, dr$ and $r\, dr = \sigma^2\, du$. Then

$$I^2 = 2\pi\sigma^2 \int_0^{\infty} \mathrm{e}^{-u}\, du = 2\pi\sigma^2.$$

Taking the positive square root (the integral is positive),

$$I = \sqrt{2\pi\sigma^2}.$$

(b) Consider

$$\int_{\mathbb{R}} \mathrm{e}^{ikx}\, \mathrm{e}^{-\frac{x^2}{2\sigma^2}}\, dx.$$

Complete the square:

$$-\frac{x^2}{2\sigma^2} + ikx = -\frac{1}{2\sigma^2}\Big(x^2 - 2i\sigma^2 k\, x\Big) = -\frac{(x - i\sigma^2 k)^2}{2\sigma^2} - \frac{\sigma^2 k^2}{2}.$$

Hence

$$\int_{\mathbb{R}} \mathrm{e}^{ikx}\, \mathrm{e}^{-\frac{x^2}{2\sigma^2}}\, dx = \mathrm{e}^{-\frac{\sigma^2 k^2}{2}} \int_{\mathbb{R}} \mathrm{e}^{-\frac{(x-i\sigma^2 k)^2}{2\sigma^2}}\, dx.$$

The integrand is an entire function with Gaussian decay, so the contour can be shifted from $\mathbb{R}$ to $\mathbb{R} + i\sigma^2 k$; the value equals the real-axis Gaussian integral from part (a):

$$\int_{\mathbb{R}} \mathrm{e}^{-\frac{(x-i\sigma^2 k)^2}{2\sigma^2}}\, dx = \int_{\mathbb{R}} \mathrm{e}^{-\frac{x^2}{2\sigma^2}}\, dx = \sqrt{2\pi\sigma^2}.$$

Therefore,

$$\int_{\mathbb{R}} \mathrm{e}^{ikx}\, \mathrm{e}^{-\frac{x^2}{2\sigma^2}}\, dx = \sqrt{2\pi\sigma^2}\, \mathrm{e}^{-\frac{\sigma^2 k^2}{2}}.$$

(c) Let C be symmetric with strictly positive eigenvalues. Then there exists an orthogonal matrix Q and a diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_d)$ with $\lambda_j > 0$ such that

$$C = Q\Lambda Q^{\top}, \qquad C^{-1} = Q\Lambda^{-1}Q^{\top}.$$

Set $y = Q^\top x$ and $q = Q^\top k$. Orthogonality gives $dx = dy$, and

$$\langle x, C^{-1}x\rangle = x^\top C^{-1}x = y^\top \Lambda^{-1} y = \sum_{j=1}^{d} \frac{y_j^2}{\lambda_j}, \qquad \langle k, x\rangle = q^\top y = \sum_{j=1}^{d} q_j y_j.$$

Thus the integral factorizes:

$$\int_{\mathbb{R}^d} \mathrm{e}^{i\langle k,x\rangle}\, \mathrm{e}^{-\frac{\langle x, C^{-1}x\rangle}{2}}\, dx = \prod_{j=1}^{d} \left(\int_{\mathbb{R}} \mathrm{e}^{iq_j y_j}\, \mathrm{e}^{-\frac{y_j^2}{2\lambda_j}}\, dy_j \right).$$

By part (b) with $\sigma^2 = \lambda_j$ and k replaced by q_j,

$$\int_{\mathbb{R}} \mathrm{e}^{iq_j y_j}\, \mathrm{e}^{-\frac{y_j^2}{2\lambda_j}}\, dy_j = \sqrt{2\pi\lambda_j}\, \mathrm{e}^{-\frac{\lambda_j q_j^2}{2}}.$$

Multiplying over j,

$$\int_{\mathbb{R}^d} \mathrm{e}^{i\langle k,x\rangle}\, \mathrm{e}^{-\frac{\langle x, C^{-1}x\rangle}{2}}\, dx = \Big(\prod_{j=1}^{d} \sqrt{2\pi\lambda_j} \Big)\, \mathrm{e}^{-\frac{1}{2}\sum_{j=1}^{d} \lambda_j q_j^2} = \sqrt{(2\pi)^d \det C}\, \mathrm{e}^{-\frac{1}{2} q^\top \Lambda q}.$$

Finally, since $q = Q^\top k$ and Q is orthogonal,

$$q^\top \Lambda q = k^\top Q \Lambda Q^\top k = k^\top C k = \langle k, Ck\rangle.$$

Hence

$$\int_{\mathbb{R}^d} \mathrm{e}^{i\langle k,x\rangle} \mathrm{e}^{-\frac{\langle x, C^{-1}x\rangle}{2}}\, dx = \sqrt{(2\pi)^d \det(C)}\, \mathrm{e}^{-\frac{\langle k, Ck\rangle}{2}}.$$

To be discussed in class: 16.10.2026