

Stochastic Differential Equations

Homework Sheet 2

Problem 1. Let $(\Omega, \mathcal{F}, P) = (\mathbb{R}_+, \mathcal{B}, \mu)$, where $\mathbb{R}_+ = [0, \infty)$, $\mathcal{B}$ denotes the Borel σ -algebra on $[0, \infty)$ and μ is a probability measure on $[0, \infty)$ with no mass on single points, i.e. $\mu(\{x\}) = 0$ for every $x \in \mathbb{R}_+$. Define

$$X_t(\omega) = \begin{cases} 1 & \text{if } t = \omega, \\ 0 & \text{otherwise,} \end{cases} \quad (1)$$

and

$$Y_t(\omega) = 0 \quad \text{for all } (t, \omega) \in \mathbb{R}_+ \times \mathbb{R}_+. \quad (2)$$

Prove that $\{X_t\}_{t \in \mathbb{R}_+}$ and $\{Y_t\}_{t \in \mathbb{R}_+}$ have the same finite-dimensional distributions and that $\{X_t\}_{t \in \mathbb{R}_+}$ is a version of $\{Y_t\}_{t \in \mathbb{R}_+}$. (Note that, nevertheless, $t \mapsto Y_t(\omega)$ is continuous for all ω and $t \mapsto X_t(\omega)$ is discontinuous for all ω).

Problem 2. Let X and Y be two independent random variables with finite variances. Show that

$$\text{var}(X + Y) = \text{var}(X) + \text{var}(Y).$$

Problem 3. Let X be a real-valued random variable with finite second moment, i.e., $E[X^2] < \infty$.

(a) Let $Y \geq 0$ be any nonnegative random variable and let $a > 0$. Show that

$$P(Y \geq a) \leq \frac{E[Y]}{a}.$$

(b) Apply part (a) with $Y = (X - E[X])^2$ and $a = \varepsilon^2$ for $\varepsilon > 0$. Conclude that

$$P(|X - E[X]| \geq \varepsilon) \leq \frac{E[(X - E[X])^2]}{\varepsilon^2}.$$

(c) Recall that $\text{var}(X) := E[(X - E[X])^2]$. Rewrite the bound from part (b) in terms of $\text{var}(X)$.

Problem 4. Let $\{B_t\}_{t \in \mathbb{R}_+}$ be 2-dimensional Brownian motion and set

$$D_\rho := \{y \in \mathbb{R}^2 : |y| < \rho\} \quad \text{for } \rho > 0, \quad (3)$$

where $|y| := \sqrt{y_1^2 + y_2^2}$. Compute $P^{x=0}(B_t \in D_\rho)$ for some fixed $t > 0$.

Problem 5. Show that:

(a) $I := \int_{\mathbb{R}} e^{-\frac{x^2}{2\sigma^2}} dx = \sqrt{2\pi\sigma^2}.$

Hint: For example, write $I^2 = \int_{\mathbb{R}^2} e^{-\frac{1}{2\sigma^2}(x^2+y^2)} dx dy$ and go to polar coordinates.

(b) $\int_{\mathbb{R}} e^{ikx} e^{-\frac{x^2}{2\sigma^2}} dx = \sqrt{2\pi\sigma^2} e^{-\frac{k^2\sigma^2}{2}}.$

(c) $\int_{\mathbb{R}^d} e^{i\langle k, x \rangle} e^{-\frac{\langle x, C^{-1}x \rangle}{2}} dx = \sqrt{(2\pi)^d \det(C)} e^{-\frac{\langle k, Ck \rangle}{2}},$ where C is a $d \times d$ symmetric matrix with strictly positive eigenvalues. Hint: Diagonalize the matrix, change variables and use the previous item.

To be discussed in class: 16.10.2026