

Stochastic Differential Equations

Homework Sheet 3 - solutions

Problem 1. Show that

$$\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} dz \{ |z|^2 \}^2 e^{-\frac{1}{2}|z|^2} = d(d+2), \quad (1)$$

where $|z|^2 := \sum_{i=1}^d (z_i)^2$. Conclude that for $\mathbb{R}$ -valued random variable $Z \sim N(0, \sigma^2)$ we have $E(Z^4) = 3\sigma^4$.

Solution. By Problem 5 of HS2

$$I(a) := \int_{\mathbb{R}^d} e^{-\frac{a}{2}|z|^2} dz = (2\pi)^{d/2} a^{-d/2}, \quad a > 0. \quad (2)$$

Differentiating under the integral,

$$I'(a) = \int_{\mathbb{R}^d} \left(-\frac{1}{2}|z|^2\right) e^{-\frac{a}{2}|z|^2} dz, \quad (3)$$

$$I''(a) = \int_{\mathbb{R}^d} \left(-\frac{1}{2}|z|^2\right)^2 e^{-\frac{a}{2}|z|^2} dz. \quad (4)$$

Hence

$$\int_{\mathbb{R}^d} (|z|^2)^2 e^{-\frac{a}{2}|z|^2} dz = 4 I''(a). \quad (5)$$

From the explicit formula for $I(a)$ above

$$I''(a) = (2\pi)^{d/2} \frac{d}{2} \left(\frac{d}{2} + 1 \right) a^{-\frac{d}{2}-2}. \quad (6)$$

Therefore

$$\int_{\mathbb{R}^d} (|z|^2)^2 e^{-\frac{a}{2}|z|^2} dz = (2\pi)^{d/2} d(d+2) a^{-\frac{d}{2}-2}. \quad (7)$$

Thus for $a = 1$ we obtain the claim.

Now if $d = 1$ and $Z \in N(0, \sigma^2)$, then $Z/\sigma \in N(0, 1)$ (see lecture notes, construction of Brownian motion via the CLT). Then

$$E[(Z/\sigma)^4] = d(d+2) = 3, \quad (8)$$

hence, $E(Z^4) = 3\sigma^4$.

Problem 2. Let $Y \sim N(0, 1)$ and set for some $a > 0$

$$Z = Y\chi_{\{|Y| \leq a\}} - Y\chi_{\{|Y| > a\}}. \quad (9)$$

Show that also $Z \sim N(0, 1)$. Also, show that $Y + Z$ is not normal, hence $X = (Y, Z)$ is not a multi-normal random variable. (In this exercise non-degenerate (multi-)normal distributions are meant).

Notation: $\chi_{\{|Y| \leq a\}}(\omega) := \chi_{\{y \in \mathbb{R} : |y| \leq a\}}(Y(\omega))$. The function $\chi_{\{|Y| > a\}}$ is defined analogously.

Solution. Step 1: Z is standard normal.

Let $\phi_Z(u) = E[e^{iuZ}]$. Then, we verify that, for any characteristic function χ_A , (10)

$$e^{iuY\chi_A} = \chi_{A^c} + e^{iuY}\chi_A, \quad (11)$$

by checking separately for $\omega \in A$ and $\omega \in A^c$. Setting $A = \{|Y| \leq a\}$ and $A^c = \{|Y| > a\}$, we have

$$e^{iuZ} = e^{iuY\chi_A} e^{-iuY\chi_{A^c}} = (\chi_{A^c} + e^{iuY}\chi_A)(\chi_A + e^{-iuY}\chi_{A^c}) = e^{iuY}\chi_A + e^{-iuY}\chi_{A^c}. \quad (12)$$

Hence,

$$\phi_Z(u) = E[e^{iuY}\chi_{\{|Y| \leq a\}}] + E[e^{-iuY}\chi_{\{|Y| > a\}}]. \quad (13)$$

Compute the second term explicitly via the density of Y :

$$E[e^{-iuY}\chi_{\{|Y| > a\}}] = \int_{|y| > a} e^{-iuy} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy. \quad (14)$$

With the change of variables $x = -y$ we get

$$\int_{|y| > a} e^{-iuy} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy = \int_{|x| > a} e^{iux} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = E[e^{iuY}\chi_{\{|Y| > a\}}]. \quad (15)$$

Hence

$$\phi_Z(u) = E[e^{iuY}\chi_{\{|Y| \leq a\}}] + E[e^{iuY}\chi_{\{|Y| > a\}}] = E[e^{iuY}] = e^{-u^2/2}. \quad (16)$$

Therefore $Z \sim N(0, 1)$.

Step 2: $Y + Z$ is not normal. By definition,

$$Y + Z = \begin{cases} 2Y, & |Y| \leq a, \\ 0, & |Y| > a. \end{cases} \quad (17)$$

If $|Y| \leq a$ then $Y \leq a$ so $2Y \leq 2a$, hence $(Y + Z) > 2a$ cannot occur in this case. If $|Y| > a$ then $Y + Z = 0 \leq 2a$, so $(Y + Z) > 2a$ also cannot occur. Therefore

$$P(Y + Z > 2a) = 0. \quad (18)$$

But any non-degenerate Gaussian random variable has an everywhere strictly positive density. Consequently, $Y + Z$ cannot be such a Gaussian.

Step 3: (Y, Z) is not jointly normal. If (Y, Z) were jointly normal (non-degenerate), then every nontrivial linear combination, in particular $Y + Z$, would be normal (non-degenerate), contradicting Step 2. Thus $X = (Y, Z)$ is not multi-normal.

Problem 3. Suppose $X_k : \Omega \rightarrow \mathbb{R}^d$ are normal for all k and $X_k \rightarrow X$ in $L^2(\Omega)$, i.e.

$$E[|X_k - X|^2] \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty. \quad (19)$$

Show that X is normal.

Solution. Write $X_k \sim N(m_k, C_k)$, where $m_k = E[X_k]$ and $C_k = \text{Cov}(X_k)$. We will pass to the limit in

$$E[e^{i\langle u, X_k \rangle}] = \exp\left(i\langle u, m_k \rangle - \frac{1}{2}\langle u, C_k u \rangle\right). \quad (20)$$

Since $X_k \rightarrow X$ in L^2 , the random vector X has finite second moments, and

$$\sup_k \|X_k\|_{L^2} < \infty. \quad (21)$$

Let $m = E[X]$ and $C = \text{Cov}(X)$.

First, by Cauchy–Schwarz,

$$|m_k - m| \leq E[|X_k - X|] \leq (E[|X_k - X|^2])^{1/2} \rightarrow 0. \quad (22)$$

Next, the second moments also converge. Indeed, for every i, j ,

$$\begin{aligned} |E[X_{k,i}X_{k,j}] - E[X_iX_j]| &\leq E[|X_{k,i} - X_i| |X_{k,j}|] \\ &\quad + E[|X_i| |X_{k,j} - X_j|] \\ &\leq \|X_{k,i} - X_i\|_{L^2} \|X_{k,j}\|_{L^2} \\ &\quad + \|X_i\|_{L^2} \|X_{k,j} - X_j\|_{L^2} \\ &\rightarrow 0. \end{aligned} \quad (23)$$

Since

$$(C_k)_{ij} = E[X_{k,i}X_{k,j}] - (m_k)_i(m_k)_j, \quad (24)$$

we conclude that $C_k \rightarrow C$ entrywise.

Finally, for every $u \in \mathbb{R}^d$, the inequality (see (28)–(31))

$$|e^{ia} - e^{ib}| \leq |a - b|, \quad a, b \in \mathbb{R}, \quad (25)$$

implies

$$\begin{aligned} |E[e^{i\langle u, X_k \rangle}] - E[e^{i\langle u, X \rangle}]| &\leq |u| E[|X_k - X|] \\ &\leq |u| \|X_k - X\|_{L^2} \rightarrow 0. \end{aligned} \quad (26)$$

Therefore, passing to the limit in the Gaussian characteristic function gives

$$\begin{aligned} E[e^{i\langle u, X \rangle}] &= \lim_{k \rightarrow \infty} \exp\left(i\langle u, m_k \rangle - \frac{1}{2}\langle u, C_k u \rangle\right) \\ &= \exp\left(i\langle u, m \rangle - \frac{1}{2}\langle u, C u \rangle\right). \end{aligned} \quad (27)$$

This is the characteristic function of $N(m, C)$. By uniqueness of characteristic functions, $X \sim N(m, C)$.

The covariance C may be singular, so the limiting Gaussian may be degenerate.

But we cannot conclude that a sequence of non-degenerate Gaussians converges to a non-degenerate Gaussian, since the inequality $>$ may become $\geq$ in the limit. Actually, such a scenario can easily happen under the assumptions of this problem: Let Z be any non-degenerate Gaussian, set $X_k = \frac{1}{k}Z$ so that it trivially converges in $L^2(\Omega)$ to $X \equiv 0$. Note that $X \equiv 0$ has the Dirac delta distribution hence it is degenerate Gaussian.

Solution (Alternative solution). Here we give a different argument: First, note that $|e^{i\langle u, x \rangle} - e^{i\langle u, y \rangle}| \leq |u| \cdot |x - y|$. This follows, e.g., from

$$|e^{i\langle u, x \rangle} - e^{i\langle u, y \rangle}| = |1 - e^{i\langle u, (x-y) \rangle}| \quad (28)$$

$$= \left| \int_0^1 \frac{d}{dw} e^{iw\langle u, (x-y) \rangle} dw \right| \quad (29)$$

$$= |i\langle u, (x-y) \rangle| \int_0^1 e^{iw\langle u, (x-y) \rangle} dw \quad (30)$$

$$\leq |u||x - y|. \quad (31)$$

Thus, we have

$$E \left[\left\{ \exp(i\langle u, X_k \rangle) - \exp(i\langle u, X \rangle) \right\}^2 \right] \leq |u|^2 E[|X_k - X|^2] \quad (32)$$

as $k \rightarrow \infty$. Therefore

$$\begin{aligned} |E[\exp(i\langle u, X_k \rangle)] - E[\exp(i\langle u, X \rangle)]| &\leq \int |\exp(i\langle u, X_k \rangle) - \exp(i\langle u, X \rangle)| dP \\ &\leq \left(\int |\exp(i\langle u, X_k \rangle) - \exp(i\langle u, X \rangle)|^2 dP \right)^{1/2} \rightarrow 0, \end{aligned} \quad (33)$$

(34)

where we used the Cauchy-Schwarz inequality. Up to now we have not used that X_k are Gaussian. Actually, we have verified, quite generally, that the L^2 -convergence implies the convergence of the characteristic function, pointwise in u .

We conclude that

$$E[\exp(i\langle u, X_k \rangle)] = \exp \left(-\frac{1}{2} \langle u, C_k u \rangle + i\langle u, m_k \rangle \right) \quad (35)$$

converge for any fixed u .

Using this, let us now justify that C_k and m_k converge to some C and m . In fact, since (35) converges, also its absolute value does, hence $\exp(-\frac{1}{2}\langle u, C_k u \rangle)$ must converge, therefore also $\langle u, C_k u \rangle$ must converge. As this holds for all u , by polarization,

$$\langle v_1, C_k v_2 \rangle = \frac{1}{4} (\langle (v_1 + v_2), C_k (v_1 + v_2) \rangle - \langle (v_1 - v_2), C_k (v_1 - v_2) \rangle) \quad (36)$$

all matrix elements converge. Now it suffices to show that convergence of $e^{i u m_k}$ for all u implies convergence of m_k . We proceed by contradiction. First, suppose $m_k \rightarrow \infty$ (perhaps along a subsequence). Set $F(u) := \lim_{k \rightarrow \infty} e^{i u m_k}$. Then, by the Riemann-Lebesgue lemma and dominated convergence, for any $f \in L^1(\mathbb{R}^d)$

$$0 = \lim_{k \rightarrow \infty} \int e^{i u m_k} f(u) du = \int F(u) f(u) du. \quad (37)$$

But F is non-zero since $F(0) = 1$ and we can choose e.g. $f(u) = \overline{F}(u) e^{-|u|^2}$ to ensure that the integral (37) is non-zero and finite, which is a contradiction. Now suppose that m_k has two (or more) finite accumulation points. Choose any $g \in L^1(\mathbb{R}^d)$ s.t. $\hat{g}$ takes different values near these accumulation points. Then

$$\int e^{i u m_k} g(u) du = \hat{g}(m_k). \quad (38)$$

By taking the limit of both sides the l.h.s. converges by dominated convergence and the r.h.s. does not as $k \mapsto \hat{g}(m_k)$ jumps between the two accumulation points. Again, a contradiction.

Since $C_k = C_k^T$ for all k , also the limiting matrix is symmetric. Similarly, since for any $u \in \mathbb{R}^d$ we have $\langle u, C_k u \rangle \geq 0$ and the inequality $\geq$ survives limits, we conclude that C has positive eigenvalues.

Problem 4. Let $\{X_n\}_{n \geq 1}$ and X be random variables on a common probability space. We say that X_n converges in probability to X , written $X_n \xrightarrow{P} X$, if for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} P(|X_n - X| > \varepsilon) = 0. \quad (39)$$

Show that if $X_n \rightarrow X$ in $L^2(\Omega)$, i.e.

$$E[(X_n - X)^2] \xrightarrow{n \rightarrow \infty} 0, \quad (40)$$

then $X_n \xrightarrow{P} X$. Hint: Use the Chebyshev's inequality from HS2, Problem 3.

Solution. To prove (b), fix $\varepsilon > 0$. By the Chebyshev's inequality,

$$P(|X_n - X| > \varepsilon) \leq \frac{E[(X_n - X)^2]}{\varepsilon^2}. \quad (41)$$

If $E[(X_n - X)^2] \rightarrow 0$ as $n \rightarrow \infty$, then the right-hand side tends to 0 for each fixed $\varepsilon > 0$, hence

$$\lim_{n \rightarrow \infty} P(|X_n - X| > \varepsilon) = 0, \quad (42)$$

which is exactly $X_n \xrightarrow{P} X$.

Side remark: The topology of convergence in probability comes from the metric

$$d_P(X, Y) = E(\min(|X - Y|, 1)) \quad (43)$$

(Ky Fan metric) The space of measurable functions equipped with this metric is denoted (L^0, d_P) . It is a complete topological vector space. However, there are no non-zero linear functionals on this space, which would be continuous in this topology (trivial topological dual), if P is non-atomic (Theorem 13.41 of C.D. Aliprantis, K.C. Border *Infinite Dimensional Analysis. A Hitchhiker's Guide*). Hence, there is also no norm generating the topology.

Problem 5. Let $\{B_t\}_{t \in \mathbb{R}_+}$ be d -dimensional Brownian motion, $d \geq 3$, starting at x . Let $K \subset \mathbb{R}^d$ be a bounded Borel set and $T > 0$. Then the random variable

$$\Omega \ni \omega \mapsto \int_0^T \chi_{\{\tilde{\omega} \in \Omega: B_t(\tilde{\omega}) \in K\}}(\omega) dt = \int_0^T \chi_K(B_t(\omega)) dt \quad (44)$$

denoted in short-hand notation

$$\int_0^T \chi_{\{B_t \in K\}} dt = \lim_{|\Pi| \rightarrow 0} \sum_{i=1}^n \chi_{\{B_{\xi_i} \in K\}} \Delta t_i \quad (45)$$

is the time this stochastic process spends in K up to the ‘time horizon’ T . (This is more clear from the Riemann sum representation, where the sum counts only the time-intervals Δt_i for which the Brownian motion B_{ξ_i} , $\xi_i \in [t_{i-1}, t_i]$, is in K). Show that

$$\lim_{T \rightarrow \infty} E^x \left[\int_0^T \chi_{\{B_t \in K\}} dt \right] = \int_K G(x, y) dy, \quad (46)$$

where $G(x, y) := \frac{c_d}{|x-y|^{d-2}}$ for some $c_d > 0$.

Solution. Let $\{B_t\}_{t \geq 0}$ be the d -dimensional Brownian motion, $d \geq 3$, starting at $x \in \mathbb{R}^d$. Recall

$$p(t, x, y) := (2\pi t)^{-d/2} e^{-\frac{|x-y|^2}{2t}}, \quad t > 0, x, y \in \mathbb{R}^d. \quad (47)$$

The expected time the process spends in K up to time T is

$$E^x \left[\int_0^T \chi_{\{B_t \in K\}} dt \right] = \int_0^T P^x(B_t \in K) dt = \int_0^T \int_K p(t, x, y) dy dt, \quad (48)$$

where the Fubini theorem justifies the exchange of integrals.

Letting $T \rightarrow \infty$ and applying the Fubini and monotone convergence theorem (for an increasing sequence of positive integrands we can exchange the limit with integral) gives

$$\lim_{T \rightarrow \infty} E^x \left[\int_0^T \chi_{\{B_t \in K\}} dt \right] = \int_K \left(\int_0^\infty p(t, x, y) dt \right) dy = \int_K G(x, y) dy, \quad (49)$$

where we defined the Green function:

$$G(x, y) = \int_0^\infty p(t, x, y) dt. \quad (50)$$

- **Computation of $G(x, y)$.** Let $r = |x - y|$. Then

$$G(x, y) = (2\pi)^{-d/2} \int_0^\infty t^{-d/2} e^{-\frac{r^2}{2t}} dt. \quad (51)$$

Using the substitution $u = \frac{r^2}{2t}$, i.e. $t = \frac{r^2}{2u}$ and $dt = -\frac{r^2}{2u^2} du$, we obtain

$$G(x, y) = (2\pi)^{-d/2} r^{2-d} \int_0^\infty 2^{\frac{d}{2}-1} u^{\frac{d}{2}-2} e^{-u} du = \frac{\Gamma(\frac{d}{2}-1)}{2\pi^{d/2}} r^{2-d}, \quad (52)$$

where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$. Equivalently,

$$G(x, y) = \frac{2}{(d-2) |S^{d-1}|} |x - y|^{2-d}, \quad (53)$$

since $|S^{d-1}| = \frac{2\pi^{d/2}}{\Gamma(d/2)}$.

- **Conclusion.** For $d \geq 3$,

$$\lim_{T \rightarrow \infty} E^x \left[\int_0^T \chi_{\{B_t \in K\}} dt \right] = \int_K G(x, y) dy = \int_K \frac{c_d}{|x - y|^{d-2}} dy, \quad (54)$$

with $c_d = \frac{\Gamma(\frac{d}{2}-1)}{2\pi^{d/2}} > 0$.

To be discussed in class: 30.10.2026