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Stochastic Differential Equations
Homework Sheet 3

This homework sheet will be discussed on 30.10.2026. The next homework sheet will be published after this date.

Problem 1. Show that

$$\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} dz\big\{|z|^2\big\}^2 \mathrm{e}^{-\frac{1}{2}|z|^2} = d(d+2), \tag{1}$$

where $|z|^2 := \sum_{i=1}^d (z_i)^2$. Conclude that for $\mathbb{R}$-valued random variable $Z \sim N(0,\sigma^2)$ we have $E(Z^4) = 3\sigma^4$.

Problem 2. Let $Y \sim N(0,1)$ and set for some $a > 0$

$$Z = Y\chi_{\{|Y|\leq a\}} - Y\chi_{\{|Y|>a\}}. \tag{2}$$

Show that also $Z \sim N(0,1)$. Also, show that $Y + Z$ is not normal, hence $X = (Y, Z)$ is not a multi-normal random variable. (In this exercise non-degenerate (multi-)normal distributions are meant).

Notation: $\chi_{\{|Y|\leq a\}}$ denotes the following characteristic function:

$$\Omega \ni \omega \mapsto \chi_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A \end{cases} \tag{3}$$

for $A := \{\,\tilde{\omega} \in \Omega : |Y(\tilde{\omega})| \leq a\,\}$. The function $\chi_{\{|Y|>a\}}$ is defined analogously.

Problem 3. Suppose $X_k : \Omega \to \mathbb{R}^d$ are normal for all k and $X_k \to X$ in $L^2(\Omega)$, i.e.

$$E[|X_k - X|^2] \to 0 \quad \text{as} \quad k \to \infty. \tag{4}$$

Show that X is normal.

Problem 4. Let $\{X_n\}_{n\geq 1}$ and X be random variables on a common probability space. We say that X_n *converges in probability* to X, written $X_n \xrightarrow{P} X$, if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0.$$

Show that if $X_n \to X$ in $L^2(\Omega)$, then $X_n \xrightarrow{P} X$. Hint: Use the Chebyshev's inequality from HS2, Problem 3.

Problem 5. Let $\{B_t\}_{t\in\mathbb{R}_+}$ be d-dimensional Brownian motion, $d \geq 3$, starting at x. Let $K \subset \mathbb{R}^d$ be a bounded Borel set and $T > 0$. Then the random variable

$$\Omega \ni \omega \mapsto \int_0^T \chi_{\{\tilde{\omega}\in\Omega : B_t(\tilde{\omega})\in K\}}(\omega)dt \tag{5}$$

denoted in short-hand notation

$$\int_0^T \chi_{\{B_t\in K\}}dt \tag{6}$$

is the time this stochastic process spends in K up to the 'time horizon' T. Show that

$$\lim_{T\to\infty} E^x\left[\int_0^T \chi_{\{B_t\in K\}}dt\right] = \int_K G(x,y)dy, \tag{7}$$

where $G(x,y) := \frac{c_d}{|x-y|^{d-2}}$ for some $c_d > 0$.

To be discussed in class: 23.10.2026