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Stochastic Differential Equations
Homework Sheet 4 - solutions

Problem 1. Let $X, Y : \Omega \to \mathbb{R}$ be two independent random variables. Show that the characteristic functions satisfy

$$\phi_{X+Y}(u) = \phi_X(u)\phi_Y(u). \tag{1}$$

Now suppose that $P(X \in F) = \int_F p_X(x)dx$ and $P(Y \in F) = \int_F p_Y(y)dy$ for $p_X, p_Y \in S(\mathbb{R})$, i.e. X, Y have Schwartz class densities p_X, p_Y. Show that

$$p_{X+Y}(z) = \int p_X(z-y)p_Y(y)dy. \tag{2}$$

Solution. If X, Y are independent, their joint law factorizes:

$$P(X \in F, Y \in G) = P(X \in F)P(Y \in G)$$

for Borel sets F, G. Now let f, g be measurable. We have, by the above,

$$\begin{aligned} P(f(X) \in F, f(Y) \in G) &= P(X \in f^{-1}(F), Y \in g^{-1}(G)) \\ &= P(X \in f^{-1}(F))P(Y \in g^{-1}(G)) \\ &= P(f(X) \in F)P(g(Y) \in G) \end{aligned} \tag{3}$$

which gives independence of $f(X)$ and $g(Y)$.

Taking $f(x) = e^{iux}$ and $g(y) = e^{iuy}$, we obtain that e^{iuX} and e^{iuY} are independent, so

$$\phi_{X+Y}(u) = E[e^{iu(X+Y)}] = E[e^{iuX}e^{iuY}] = E[e^{iuX}]\,E[e^{iuY}] = \phi_X(u)\,\phi_Y(u). \tag{4}$$

We use the Fourier transform convention

$$\widehat{f}(u) = \int_{\mathbb{R}} e^{iux} f(x)\,dx, \qquad f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iux}\,\widehat{f}(u)\,du.$$

If X, Y have densities $p_X, p_Y \in L^1(\mathbb{R})$, then

$$\phi_X(u) = \widehat{p_X}(u), \qquad \phi_Y(u) = \widehat{p_Y}(u).$$

From (4) we have

$$\widehat{p_{X+Y}}(u) = \phi_{X+Y}(u) = \phi_X(u)\,\phi_Y(u) = \widehat{p_X}(u)\,\widehat{p_Y}(u).$$

By the convolution theorem and the invertibility of the Fourier transform,

$$p_{X+Y} = p_X * p_Y, \qquad p_{X+Y}(z) = \int_{\mathbb{R}} p_X(z-y)\,p_Y(y)\,dy.$$

Problem 2. Let $\{B_t\}_{t\in\mathbb{R}_+}$ be the d-dimensional Brownian motion starting at zero and $v \in \mathbb{R}^d$ a fixed non-zero vector. Compute the distribution of $X_t^{(v)} := \langle v, (B_{t+1} - B_t)\rangle$.

Solution. We have

$$X_t^{(v)} = \sum_{i=1}^{d} v_i (B_{t+1} - B_t)_i. \tag{5}$$

Since the Brownian motion is a multi-normal random variable, we know from the lecture that $X_t^{(v)}$ is a Gaussian random variable for each t with $E(X_t^{(v)}) = 0$, since $E(B_s) = 0$. Thus $X_t^{(v)} \sim N(0, \sigma^2)$. We just have to compute the variance σ^2.

We know from the lecture that

$$E((B_t)_i (B_s)_j) = \min(t, s)\delta_{i,j} = 0 \text{ for } i \neq j. \tag{6}$$

Consequently, remembering that $E(B_s) = 0$, we see that different summands in (5) are uncorrelated. Thus, by a small generalization of arguments from HS2, Problem 2, we have

$$\operatorname{var}(X_t^{(v)}) = \sum_{i=1}^{d} \operatorname{var}(v_i (B_{t+1} - B_t)_i). \tag{7}$$

We know from the lecture that $(B_{t+1} - B_t)_i \sim N(0, 1)$. Hence,

$$v_i (B_{t+1} - B_t)_i \sim N(0, v_i^2). \tag{8}$$

(See justification below). In other words,

$$\operatorname{var}(v_i (B_{t+1} - B_t)_i) = v_i^2 \tag{9}$$

and by (7),

$$\operatorname{var}(X_t^{(v)}) = \sum_{i=1}^{d} v_i^2 = |v|^2, \tag{10}$$

which is independent of t.

It remains to justify (8). Actually we have seen it already in the lecture, in the context of random walks, but it is even easier to see it using the characteristic function. Suppose $Y \sim N(0, \sigma^2)$ and $a > 0$. Then

$$E(\mathrm{e}^{iuY}) = \mathrm{e}^{-\frac{1}{2}\sigma^2 u^2}. \tag{11}$$

Hence

$$E(\mathrm{e}^{iu(aY)}) = \mathrm{e}^{-\frac{1}{2}(\sigma^2 a^2) u^2}. \tag{12}$$

Thus, if $\operatorname{var}(Y) = \sigma^2$ then $\operatorname{var}(aY) = a^2\sigma^2$.

To be discussed in class: 06.11.2026 (Together with some comments to HS3).