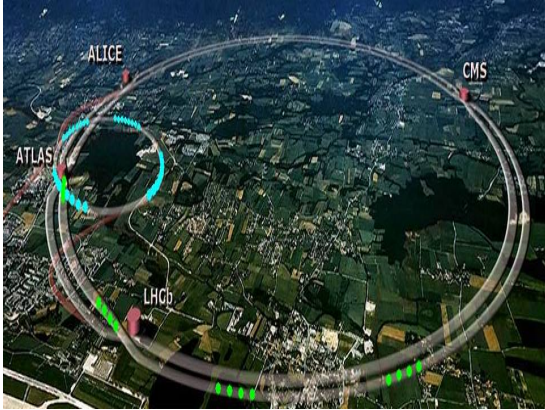


NCN Project ‘Interaction in quantum field theory’

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Imagine an electron colliding with a positron in a particle accelerator. What is the probability that a Higgs particle will be produced in this collision?



The theoretical framework physicists use to answer such questions is called quantum field theory (QFT). The usual computational method within this framework gives an answer in the form of an infinite sum of successive corrections. By considering only first few terms of this sum, one often obtains a very good agreement with experiments. However, if we could sum up all the terms, the result would most likely be infinity. Can we claim agreement of theory and experiment in this situation?

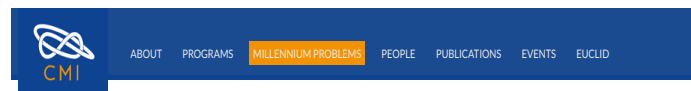
In the light of the above dilemma, the problem of logical consistency of interacting QFT in four-dimensional spacetime remains open to date. This fact is at the basis of the *Yang-Mills Existence and Mass-Gap Millennium*

Problem of the Clay Mathematics Institute which concerns a rigorous mathematical construction of the Yang-Mills theories.

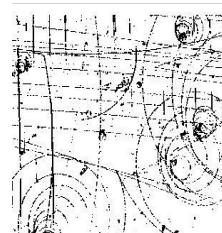
The physical context of this problem is clear: these QFTs are the main building blocks of the Standard Model of particle physics. But on the mathematical side it is legitimate to ask what actually should be constructed.

To answer such questions, already in the 1950s the first axiomatic systems for QFT have been formulated. They give a precise mathematical meaning to very general principles such as stability of matter, independence of a choice of an inertial reference frame or mutual independence of measurements performed in spacelike separated regions. A good news was that such general assumptions suffice to consistently define probabilities of collision processes. However, natural assumptions ensuring the presence of non-trivial physical processes, involving interaction between particles or particle production, are not known. It is also not known, after five decades of research, if there is any interacting QFT in four-dimensional spacetime, fitting into the axiomatic settings mentioned above.

While we are not going to construct the Yang-Mills theories, the goal of this project is to study the problem of interaction in axiomatic QFT and to explore some new approaches to construction of interacting models.



Yang-Mills and Mass Gap



The laws of quantum physics stand to the world of elementary particles in the way that Newton's laws of classical mechanics stand to the macroscopic world. Almost half a century ago, Yang and Mills introduced a remarkable new framework to describe elementary particles using structures that also occur in geometry. Quantum Yang-Mills theory is now the foundation of most of elementary particle theory, and its predictions have been tested at many experimental laboratories, but its mathematical foundation is still unclear. The successful use of Yang-Mills theory to describe the strong interactions of elementary particles depends on a subtle quantum mechanical property called the "mass gap": the quantum particles have positive masses, even though the classical waves travel at the speed of light. This property has been discovered by physicists from experiment and confirmed by computer simulations, but it still has not been understood from a theoretical point of view. Progress in establishing the existence of the Yang-Mills theory and a mass gap will require the introduction of fundamental new ideas both in physics and in mathematics.

This problem is: Unsolved

Rules:

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Related Documents:

[Official Problem Description](#)

[Status of the Problem by Michael Douglas](#)

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[Lecture by Lorenzo Sadun](#)

Objective (A) of this project was to exploit an unexpected relation between interaction in QFT and a non-ergodic behaviour of a certain auxiliary system. Ergodicity means that averages over time of certain physical quantities are equal to averages over a time-independent statistical ensemble. On the side of QFT, this time-independence means that configurations of particles do not change in physical processes, thus there is no interaction. By proving suitable *ergodic theorems*, we wanted to restrict the realm of interacting theories. This investigation led Janik Kruse to applying the Mourre theory for the first time in the setting of axiomatic QFT and proving the convergence of a single Araki-Haag particle detector [Kr24a, Kr24b]. While this is not directly relevant to the problem of interaction, it brings us closer to proving asymptotic completeness in QFT. Regarding the original problem of interaction, we found something unexpected: massless free field in two dimensional spacetime, considered so far a non-interacting model, has non-trivial scattering amplitudes due to infrared problems! [DM22]

Objective (B) was to construct a class of QFT in two-dimensional spacetime, which arise by restricting an N -element family of free fields to the unit sphere. These so called $O(N)$ non-linear sigma models have many properties in common with the Yang-Mills theories in particular they are asymptotically free. We took significant steps towards a construction of such models by the Balaban method, by working out the exponential decay of lattice Green functions [DST24a] and the Balaban variational problem [DST24b]. Moreover, using the Polchinski equation, Paweł Duch found a new construction of the Gross-Neveu model, which is also an asymptotically free theory [Du24]. Furthermore, we explored the technique of stochastic quantization and obtained interesting methodological advances in the realm of polynomial models [DGR23, Du24]. In particular, in [DDJ25] the axioms of rotational invariance and reflection positivity are verified simultaneously by constructing the model first on a sphere and then taking the radius of the sphere to infinity.

Objective (C) was to look at other QFT which arise by imposing constraints on free theories and search for interacting models in four-dimensional spacetime within this unexplored class. More generally, we studied scalar Euclidean quantum field theories with interactions defined by bounded, measurable functions. With the Heaviside theta function in mind, imposing the positivity constraint on the field, we assumed that these functions have well-defined limits at both positive and negative infinity. In the spirit of the Buchholz-Verch renormalization we started from a large class of field renormalization functions and looked for choices admitting an interesting UV behaviour. In [Dy25], we exhibited choices for which all the connected Schwinger functions, except perhaps for the two-point function, exist non-perturbatively in the UV limit. Moreover, we obtained closed form formulas for these Schwinger functions demonstrating non-triviality of the field. These results hold in arbitrary spacetime dimension $d \geq 2$. The problem of controlling also the two-point function is currently under investigation.

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Sources of the figures: <http://scienceblogs.com>, <http://www.claymath.org>.