Prof. UAM dr hab. Wojciech Dybalski

AMU Poznań

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Stochastic Differential Equations
Homework Sheet 1

Problem 1. Let P be a probability measure. Suppose $P(A) = \frac{3}{4}$ and $P(B) = \frac{1}{3}$. Show that

$$\frac{1}{12} \; \leq \; P(A \cap B) \; \leq \; \frac{1}{3}.$$

Problem 2. Let Ω be an infinite set (countable or not) and let

$$\mathcal{G} \; = \; \{A \subseteq \Omega : \; A \text{ is finite}\} \; \cup \; \{A \subseteq \Omega : \; \Omega \setminus A \text{ is finite}\}$$

be the family of all finite or cofinite subsets of Ω. Show that $\mathcal{G}$ is an algebra of sets but not a σ-algebra.

Problem 3. Let $\{H_1, \dots, H_k\}$ be a finite partition of Ω with $P(H_i) > 0$ for all i.

(a) Prove the *law of total probability*:

$$P(B) = \sum_{i=1}^{k} P(B \mid H_i)\, P(H_i).$$

(b) Use this result to derive the following formula

$$P(H_j \mid B) = \frac{P(B \mid H_j)\, P(H_j)}{\sum_{i=1}^{k} P(B \mid H_i)\, P(H_i)}.$$

under the assumption $P(B) \neq 0$.

Problem 4. A disease affects 1% of a population. A test for the disease has:

- sensitivity $P(+ \mid D) = 0.99$ (true positive rate). That is, a person who has the disease will have a positive test result with probability 0.99.
- specificity $P(- \mid \overline{D}) = 0.95$ (true negative rate). That is, a person who does not have the disease will have a negative test result with probability 0.95.

If a randomly chosen person tests positive, what is $P(D \mid +)$, i.e., the probability that a positively tested person has the disease? Hint: Use Problem 3.

Problem 5. A box contains two coins:

- Coin A is fair: $P(H \mid A) = \frac{1}{2}$,
- Coin B is biased: $P(H \mid B) = \frac{3}{4}$.

One coin is chosen at random (so $P(A) = P(B) = \frac{1}{2}$) and tossed.

(a) The first toss is heads. Compute $P(B \mid H)$.

(b) The coin is tossed again and the second toss is also heads. Compute $P(B \mid HH)$.

(c) (Generalization) After n heads in a row, express $P(B \mid H^n)$ in closed form.

Problem 6. Let $(\Omega, \mathcal{F}, P)$ be a probability space.

(a) (**Continuity from below**) Let $(A_n)_{n\geq 1}$ be an increasing sequence of events ($A_n \subseteq A_{n+1}$ for all n). Show that

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n).$$

(b) (**Continuity from above**) Let $(B_n)_{n\geq 1}$ be a decreasing sequence of events ($B_{n+1} \subseteq B_n$ for all n). Show that

$$P\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} P(B_n).$$

Problem 7. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(A_n)_{n\geq 1}$ a sequence of events. Recall the *limsup* ("infinitely often") event

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \{\omega : \ \omega \in A_n \text{ for infinitely many } n\}.$$

Prove:

(i) If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then

$$P\left(\limsup_{n\to\infty} A_n\right) = 0.$$

(ii) If the events (A_n) are independent and $\sum_{n=1}^{\infty} P(A_n) = \infty$, then

$$P\left(\limsup_{n\to\infty} A_n\right) = 1.$$

Hint for (ii): Use independence to write $P\big(\bigcap_{k=n}^{m} A_k^c\big) = \prod_{k=n}^{m}(1 - P(A_k))$ and the inequality $1 - x \leq e^{-x}$ for $x \in [0, 1]$.

To be discussed in class: 10.10.2025