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Stochastic Differential Equations
Homework Sheet 10 - solutions

Problem 1. Let $X \in L^1(\Omega, \mathcal{F}, P)$ and $\mathcal{H} \subset \mathcal{F}$ be a σ-field. Then

$$X \geq 0 \quad \Rightarrow \quad E[X|\mathcal{H}] \geq 0, \tag{1}$$

where both inequalities hold P-almost surely.

Solution. Recall the defining property:

$$\int_H E[X|\mathcal{H}] dP = \int_H X dP \tag{2}$$

for all $H \in \mathcal{H}$. The set $N_n = \{ \omega \in \Omega \,|\, E[X|\mathcal{H}](\omega) < -1/n \}$ is in $\mathcal{H}$. Suppose it is of non zero measure. Then

$$0 > \int_{N_n} (-1/n) dP \geq \int_{N_n} E[X|\mathcal{H}] dP = \int_{N_n} X dP \geq 0, \tag{3}$$

which is a contradiction. Thus $P(N_n) = 0$ and $P(\bigcup_n N_n) = 0$ but $\bigcup_n N_n = \{ \omega \in \Omega \,|\, E[X|\mathcal{H}](\omega) < 0 \}$.

Problem 2. (Jensen inequality) Let $X \in L^1(\Omega, \mathcal{F}, P)$ and $\mathcal{H} \subset \mathcal{F}$ be a σ-field. Then, if $g : \mathbb{R} \to \mathbb{R}$ is convex and $E[|g(X)|] < \infty$, then

$$g(E[X|\mathcal{H}]) \leq E[g(X)|\mathcal{H}]. \tag{4}$$

Hint: You can use the fact that if $g : \mathbb{R} \to \mathbb{R}$ is convex then $g = \sup_n (a_n x + b_n)$ for a countable collection of real numbers $(a_n, b_n)_{n \in \mathbb{N}}$. Also, Problem 1 may be useful.

Solution. We have the following, by linearity,

$$g(E[X|\mathcal{H}]) = \sup_n (a_n E[X|\mathcal{H}] + b_n) = \sup_n E[a_n X + b_n|\mathcal{H}]. \tag{5}$$

Now, by Problem 1 and linearity, we have that $E[Z_1|\mathcal{H}] \leq E[Z_2|\mathcal{H}]$ for $Z_1 \leq Z_2$, $Z_1, Z_2 \in L^1(\Omega, \mathcal{F}, P)$. Hence,

$$E[a_n X + b_n|\mathcal{H}] \leq E[\sup_n (a_n X + b_n)|\mathcal{H}] = E[g(X)|\mathcal{H}]. \tag{6}$$

Side remark on the hint: Since g is convex on the open set $\mathbb{R}$, it is also continuous. Then the region above the graph $G = \{ (x, y) \in \mathbb{R}^2 : y \geq g(x) \}$ is a convex set which is closed (inverse image of a closed set). Fact: any closed convex set is an intersection of half-planes $H_n = \{(x, y) \in \mathbb{R}^2 : y \geq a_n x + b_n\}$, i.e., $G = \bigcap_n H_n$. By drawing a picture it is plausible that $g = \sup_n (a_n x + b_n)$.

Problem 3. Let $\{B_s\}_{s\geq 0}$ be a standard Brownian motion with its natural filtration $\{\mathcal{F}_s\}_{s\geq 0}$, that is $\mathcal{F}_s = \sigma(B_u : 0 \leq u \leq s)$. Define the process $\{N_s\}_{s\geq 0}$ by

$$N_s = B_s^3 - 3sB_s, \quad s \geq 0.$$

Show that $\{N_s\}_{s\geq 0}$ is a martingale with respect to $\{\mathcal{F}_s\}_{s\geq 0}$.

Solution. Property (i) from the definition of the martingale follows from the fact that B_s, hence its powers, are measurable w.r.t. $\mathcal{F}_s$. Property (ii) follows from the fact that B_s is a Gaussian random variable hence has finite moments. Regarding (iii), we must show for $t < s$

$$E[N_s \mid \mathcal{F}_t] = N_t \quad \text{a.s.}$$

Let $X := B_s - B_t$. Then X is independent of $\mathcal{F}_t$ and has distribution $N(0, s-t)$. We use the decomposition

$$B_s = B_t + X.$$

(i) Expand

$$B_s^3 = (B_t + X)^3 = B_t^3 + 3B_t^2 X + 3B_t X^2 + X^3.$$

(ii) Using independence of X and $\mathcal{F}_t$, and the moments

$$E[X] = 0, \quad E[X^2] = s - t, \quad E[X^3] = 0,$$

we obtain

$$E[B_s^3 \mid \mathcal{F}_t] = B_t^3 + 3B_t^2 E[X] + 3B_t E[X^2] + E[X^3] = B_t^3 + 3B_t(s-t).$$

(iii) For the second term in N_s, note that s is deterministic (a constant function of ω), so

$$E[sB_s \mid \mathcal{F}_t] = s\,E[B_s \mid \mathcal{F}_t].$$

From $B_s = B_t + X$ and $E[X \mid \mathcal{F}_t] = 0$, we get

$$E[B_s \mid \mathcal{F}_t] = B_t,$$

hence

$$E[sB_s \mid \mathcal{F}_t] = sB_t.$$

(iv) Combine the two parts:

$$E[N_s \mid \mathcal{F}_t] = E[B_s^3 - 3sB_s \mid \mathcal{F}_t] = (B_t^3 + 3B_t(s-t)) - 3sB_t.$$

Simplifying,

$$B_t^3 + 3B_t s - 3B_t t - 3sB_t = B_t^3 - 3tB_t = N_t.$$

Thus for all $0 \leq t < s$,

$$E[N_s \mid \mathcal{F}_t] = N_t \quad \text{a.s.}$$

Therefore $\{N_s\}_{s\geq 0}$ is a martingale with respect to $\{\mathcal{F}_s\}_{s\geq 0}$.

Problem 4. Let $\{X_n\}_{n\geq 1}$ be a sequence of independent random variables, taking values in $\{-1, 1\}$, with

$$P(X_n = 1) = P(X_n = -1) = \tfrac{1}{2},$$

and let $\{\mathcal{F}_n\}_{n\geq 0}$ be the natural filtration, that is $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$. A betting strategy is a sequence $\{H_n\}_{n\geq 1}$ of random variables such that each H_n is $\mathcal{F}_{n-1}$-measurable (the stake at time n depends only on the past). Define the capital process $\{K_n\}_{n\geq 0}$ by

$$K_0 \in \mathbb{R}, \qquad K_n = K_0 + \sum_{k=1}^{n} H_k X_k, \quad n \geq 1.$$

Assume that $E[|K_0|], E[|H_k X_k|] < \infty$ for each $k \in \mathbb{N}$.

(a) Show that $\{K_n\}_{n\geq 0}$ is a martingale with respect to $\{\mathcal{F}_n\}_{n\geq 0}$,

(b) How does the 'doubling strategy', described in the lecture below the definition of martingales, fits into this framework?

Solution. (a) We check the martingale property for $\{K_n\}_{n\geq 0}$.

(i) Adaptedness: For each n,

$$K_n = K_0 + \sum_{k=1}^{n} H_k X_k$$

is $\mathcal{F}_n$-measurable, since H_k is $\mathcal{F}_{k-1}$-measurable and X_k is $\mathcal{F}_k$-measurable, hence $H_k X_k$ is $\mathcal{F}_k$- and thus $\mathcal{F}_n$-measurable.

(ii) Integrability: If each $H_k X_k$ is integrable, then K_n is integrable as a finite sum of integrable random variables.

(iii) Martingale property: For $n \geq 0$,

$$E[K_{n+1} \mid \mathcal{F}_n] = E\big(K_n + H_{n+1}X_{n+1} \mid \mathcal{F}_n\big) = K_n + E\big(H_{n+1}X_{n+1} \mid \mathcal{F}_n\big).$$

Here H_{n+1} is $\mathcal{F}_n$-measurable and X_{n+1} is independent of $\mathcal{F}_n$ with $E[X_{n+1}] = 0$, so

$$E\big(H_{n+1}X_{n+1} \mid \mathcal{F}_n\big) = H_{n+1}E(X_{n+1} \mid \mathcal{F}_n) = H_{n+1}E(X_{n+1}) = 0.$$

Hence

$$E[K_{n+1} \mid \mathcal{F}_n] = K_n.$$

Thus $\{K_n\}_{n\geq 0}$ is a martingale with respect to $\{\mathcal{F}_n\}_{n\geq 0}$.

(b) The 'doubling strategy' fits in the above framework as follows:

$$H_1(\omega) = 1,$$

and for $n \geq 1$,

$$H_{n+1}(\omega) = \begin{cases} 2\,H_n(\omega), & \text{if } X_n(\omega) = -1, \\ 1, & \text{if } X_n(\omega) = +1. \end{cases}$$

Important thing is that H_{n+1} depends only on $X_1, \dots X_n$, i.e., is measurable w.r.t. $\mathcal{F}_n$.

Problem 5. Let $X \in L^1(\Omega, \mathcal{F}, P)$ be a random variable and let $\{\mathcal{H}_t\}_{t\in\mathbb{R}_+}$ be a filtration in $\mathcal{F}$. Show that $X_t = E[\,X\,|\mathcal{H}_t]$ is a martingale w.r.t. $\{\mathcal{H}_t\}_{t\in\mathbb{R}_+}$.

Solution. We check the properties from the definition of a martingale:

(i) X_t is $\mathcal{H}_t$-measurable by definition of the conditional expectation.

(ii) By the Jensen inequality (4) with $g(x) = |x|$

$$E\big[|X_t|\big] = E\big[|E[\,X\,|\mathcal{H}_t]|\big] \le E\big[E[\,|X|\,|\mathcal{H}_t]\big] = E[|X|] < \infty. \tag{7}$$

(iii) By the tower property: For $s > t$ we have $\mathcal{H}_s \supset \mathcal{H}_t$ thus

$$E[X_s|\mathcal{H}_t] = E[E[\,X\,|\mathcal{H}_s]|\mathcal{H}_t] = E[\,X\,|\mathcal{H}_t]. \tag{8}$$

Side remark: Recall the definition of $E[X|\mathcal{H}]$ for finite $\mathcal{H}$. We interpreted it as a step function which is best approximating X. This exercise shows that a sequence $\{X_t\}_{t\in\mathbb{R}_+}$ of such approximations with step functions over finer and finer intervals is a martingale. This is a rather different intuition behind the martingale than gambling strategies.

To be discussed in class: 9.01.2026