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Stochastic Differential Equations
Homework Sheet 10

Problem 1. Let $X \in L^1(\Omega, \mathcal{F}, P)$ and $\mathcal{H} \subset \mathcal{F}$ be a σ-field. Then

$$X \geq 0 \quad \Rightarrow \quad E[X|\mathcal{H}] \geq 0, \tag{1}$$

where both inequalities hold P-almost surely.

Problem 2. (Jensen inequality) Let $X \in L^1(\Omega, \mathcal{F}, P)$ and $\mathcal{H} \subset \mathcal{F}$ be a σ-field. Then, if $g : \mathbb{R} \to \mathbb{R}$ is convex and $E[|g(X)|] < \infty$, then

$$g(E[X|\mathcal{H}]) \leq E[g(X)|\mathcal{H}]. \tag{2}$$

Hint: You can use the fact that if $g : \mathbb{R} \to \mathbb{R}$ is convex then $g = \sup_n(a_n x + b_n)$ for a countable collection of real numbers $(a_n, b_n)_{n\in\mathbb{N}}$. Also, Problem 1 may be useful.

Problem 3. Let $\{B_s\}_{s\geq 0}$ be a standard Brownian motion with its natural filtration $\{\mathcal{F}_s\}_{s\geq 0}$, that is $\mathcal{F}_s = \sigma(B_u : 0 \leq u \leq s)$. Define the process $\{N_s\}_{s\geq 0}$ by

$$N_s = B_s^3 - 3sB_s, \quad s \geq 0.$$

Show that $\{N_s\}_{s\geq 0}$ is a martingale with respect to $\{\mathcal{F}_s\}_{s\geq 0}$.

Problem 4. Let $\{X_n\}_{n\geq 1}$ be a sequence of independent random variables, taking values in $\{-1, 1\}$, with

$$P(X_n = 1) = P(X_n = -1) = \tfrac{1}{2},$$

and let $\{\mathcal{F}_n\}_{n\geq 0}$ be the natural filtration, that is $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

A betting strategy is a sequence $\{H_n\}_{n\geq 1}$ of random variables such that each H_n is $\mathcal{F}_{n-1}$-measurable (the stake at time n depends only on the past). Define the capital process $\{K_n\}_{n\geq 0}$ by

$$K_0 \in \mathbb{R}, \qquad K_n = K_0 + \sum_{k=1}^{n} H_k X_k, \quad n \geq 1.$$

Assume that $E[|K_0|], E[|H_k X_k|] < \infty$ for each $k \in \mathbb{N}$.

(a) Show that $\{K_n\}_{n\geq 0}$ is a martingale with respect to $\{\mathcal{F}_n\}_{n\geq 0}$.

(b) How does the 'doubling strategy', described in the lecture below the definition of martingales, fits into this framework?

Problem 5. Let $X \in L^1(\Omega, \mathcal{F}, P)$ be a random variable and let $\{\mathcal{H}_t\}_{t\in\mathbb{R}_+}$ be a filtration in $\mathcal{F}$. Show that $X_t = E[\,X\,|\mathcal{H}_t]$ is a martingale w.r.t. $\{\mathcal{H}_t\}_{t\in\mathbb{R}_+}$.

To be discussed in class: 9.01.2026