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Stochastic Differential Equations
Homework Sheet 11 - solutions

Problem 1. Recall the following definition:

Definition 0.1. Let $\mathcal{V}_{\mathcal{H}}(S,T)$ be the class of functions

$$(t,\omega) \mapsto f(t,\omega) : \mathbb{R}_+ \times \Omega \to \mathbb{R} \tag{1}$$

such that

(i) f is $\mathcal{B}(\mathbb{R}_+) \times \mathcal{F}$-measurable.

(ii') There exists a filtration $\{\mathcal{H}_t\}_{t\in\mathbb{R}_+}$ s.t. $(B_t, \mathcal{H}_t)_{t\in\mathbb{R}_+}$ is a martingale and f is $\{\mathcal{H}_t\}_{t\in\mathbb{R}_+}$-adapted.

(iii) $E\big[\int_S^T f(t,\,\cdot\,)^2 dt\big] < \infty$.

In the context of this definition a function $\varphi \in \mathcal{V}_{\mathcal{H}}(S,T)$ is called elementary if it has the form

$$\varphi(t,\omega) = \sum_{j\geq 0} e_j(\omega)\chi_{[t_j,t_{j+1})}(t), \tag{2}$$

where e_j is $\mathcal{H}_{t_j}$-measurable. Prove the following generalization of the Itô lemma:

Lemma 0.2. If $\varphi \in \mathcal{V}_{\mathcal{H}}(S,T)$ is elementary, then

$$E\bigg[\bigg(\int_S^T \varphi(t,\,\cdot\,)dB_t(\,\cdot\,)\bigg)^2\bigg] = E\bigg[\int_S^T \varphi(t,\,\cdot\,)^2 dt\bigg]. \tag{3}$$

Hint: You can use without proof, that for any filtration $\{\mathcal{H}_t\}_{t\in\mathbb{R}_+}$, for which $(B_t,\mathcal{H}_t)_{t\in\mathbb{R}_+}$ is a continuous martingale, $(B_t^2 - t, \mathcal{H}_t)_{t\in\mathbb{R}_+}$ is also a martingale (cf. Baldi "Stochastic calculus", Theorem 5.16, Example 5.4).

Solution. Put $B_j := B_{t_j}$, $\Delta B_j := B_{j+1} - B_j$ and $\mathcal{H}_j := \mathcal{H}_{t_j}$ as shorthand notations. Then, we have

$$E[e_i e_j \Delta B_i \Delta B_j] = \begin{cases} 0 & \text{if} \quad i \neq j, \\ E[e_j^2](t_{j+1} - t_j) & \text{if} \quad i = j. \end{cases} \tag{4}$$

Let us justify the first line, supposing that $i < j$. To this end, we first show in steps that $e_i e_j \Delta B_i$ is $\mathcal{H}_j$-measurable:

- By adaptedness, e_j is $\mathcal{H}_j$-measurable.
- e_i is $\mathcal{H}_i$-measurable, hence $\mathcal{H}_j$-measurable, since $\mathcal{H}_i \subset \mathcal{H}_j$.
- Regarding $\Delta B_i = B_{i+1} - B_i$, since $(B_t, \mathcal{H}_t)_{t\in\mathbb{R}_+}$ is a martingale, B_i is $\mathcal{H}_i$-measurable. Since $i+1 \leq j$, we have that B_{i+1} is $\mathcal{H}_j$ measurable. So ΔB_i is $\mathcal{H}_j$-measurable.

Now that we know, that $e_i e_j \Delta B_i$ is $\mathcal{H}_j$-measurable, let us write, using the tower property

$$\begin{aligned} E[e_i e_j \Delta B_i \Delta B_j] &= E\big[E[e_i e_j \Delta B_i \Delta B_j | \mathcal{H}_j]\big] \\ &= E\big[e_i e_j \Delta B_i E[\Delta B_j | \mathcal{H}_j]\big], \end{aligned} \tag{5}$$

where in the last step we used the property

$$E[XY|\mathcal{G}] = XE[Y|\mathcal{G}] \quad \text{if } X \text{ is } \mathcal{G}\text{-measurable.} \tag{6}$$

Now we check that $E[\Delta B_j | \mathcal{H}_j] = E[B_{j+1}|\mathcal{H}_j] - E[B_j|\mathcal{H}_j] = 0$. In fact, by the martingale property of (ii'),

$$E[B_{j+1}|\mathcal{H}_j] = B_j = E[B_j|\mathcal{H}_j], \tag{7}$$

where in the last step we used again (6). Thus we checked the first line in (4).

Regarding the second line of (4), we write, analogously to (5):

$$\begin{aligned} E[e_j^2(\Delta B_j)^2] &= E\big[E[e_j^2(\Delta B_j)^2|\mathcal{H}_j]\big] \\ &= E\big[e_j^2 E[(\Delta B_j)^2|\mathcal{H}_j]\big]. \end{aligned} \tag{8}$$

Now we use the martingale property of (ii') and from the hint to compute $E[(\Delta B_j)^2|\mathcal{H}_j]$:

$$\begin{aligned} E[(\Delta B_j)^2|\mathcal{H}_j] &= E[B_{j+1}^2 - 2B_{j+1}B_j + B_j^2|\mathcal{H}_j] \\ &= E[B_{j+1}^2|\mathcal{H}_j] - 2B_j E[B_{j+1}|\mathcal{H}_j] + B_j^2 \\ &= t_{j+1} + E[(B_{j+1}^2 - t_{j+1})|\mathcal{H}_j] - 2B_j E[B_{j+1}|\mathcal{H}_j] + B_j^2 \\ &= t_{j+1} - t_j, \end{aligned} \tag{9}$$

where in the second step we used (6) and in the last step the hint and the martingale property. This gives the second line in (4).

Now we complete the proof like in the version of the lemma for $\mathcal{V}(S,T)$. On the one hand,

$$E\left[\left(\int_S^T \varphi dB\right)^2\right] = \sum_{i,j} E[e_i e_j \Delta B_i \Delta B_j] = \sum_j E[e_j^2](t_{j+1} - t_j). \tag{10}$$

On the other hand

$$\begin{aligned} E\left[\int_S^T \varphi(t,\cdot)^2 dt\right] &= E\left[\sum_{j\geq 0} \int_S^T |e_j|^2 \chi_{[t_j,t_{j+1})}(t)dt\right] \\ &= E\left[\sum_{j\geq 0} |e_j|^2 (t_{j+1} - t_j)\right] = \sum_{j\geq 0} E(|e_j|^2)(t_{j+1} - t_j). \end{aligned} \tag{11}$$

Problem 2. Let $(\Omega, \{\mathcal{H}_t\}_{t\in\mathcal{T}}, \mathcal{F}, P)$ be a filtered probability space. Let $\tau : \Omega \to \overline{\mathcal{T}}$ be a stopping time, i.e. $\{\tau \leq t\} \subset \mathcal{H}_t$ for any $t \in \mathcal{T}$.

(a) Show that for any $s \in \mathcal{T}$ also $\tau(\cdot) \wedge s := \min(\tau(\cdot), s)$ is a stopping time.

(b) Show that the stopping time $\tau \wedge s$ is bounded by s. That is, $P(\tau \wedge s \leq s) = 1$.

Solution. Regarding (a), consider the set

$$\begin{aligned} A_t &:= \{\,\omega \in \Omega \,:\, \tau(\omega) \wedge s \leq t\} \\ &= \{\,\omega \in \Omega \,:\, \tau(\omega) \leq s, \tau(\omega) \wedge s \leq t\} \cup \{\,\omega \in \Omega \,:\, \tau(\omega) > s, \tau(\omega) \wedge s \leq t\} \\ &= \{\,\omega \in \Omega \,:\, \tau(\omega) \leq s, \tau(\omega) \leq t\} \cup \{\,\omega \in \Omega \,:\, \tau(\omega) > s, s \leq t\} \\ &= \{\,\omega \in \Omega \,:\, \tau(\omega) \leq s \wedge t\} \cup \{\,\omega \in \Omega \,:\, \tau(\omega) > s, s \leq t\}. \end{aligned} \tag{12}$$

First, suppose that $t < s$. Then

$$A_t := \{\,\omega \in \Omega \,:\, \tau(\omega) \leq t\} \in \mathcal{H}_t, \tag{13}$$

since τ is a stopping time. Now suppose $t \geq s$. Then

$$A_t = \{\,\omega \in \Omega \,:\, \tau(\omega) \leq s\} \cup \{\,\omega \in \Omega \,:\, \tau(\omega) > s\} = \Omega \in \mathcal{H}_t, \tag{14}$$

by definition of the σ-algebra.

As for (b), we have $\{\tau \wedge s \leq s\} = A_s = \Omega$ by (14). Hence $P(\tau \wedge s \leq s) = 1$.

Problem 3. Let $X \in L^\infty(\Omega, \mathcal{F}, P)$ and $Y \in L^1(\Omega, \mathcal{F}, P)$. Show that

$$E[X \cdot Y|\mathcal{H}] = X \cdot E[Y|\mathcal{H}], \tag{15}$$

if X is $\mathcal{H}$-measurable. (Recall that in the lecture we covered $X \in L^\infty(\Omega, \mathcal{F}, P), Y \in L^2(\Omega, \mathcal{F}, P)$ and $X, Y \in L^2(\Omega, \mathcal{F}, P)$.)

Solution. By the defining property of conditional expectations and considering that $XY \in L^1(\Omega, \mathcal{F}, P)$

$$\int_H E[YX|\mathcal{H}]dP = \int_H XY dP. \tag{16}$$

On the other hand

$$\begin{aligned} \int_H YE[X|\mathcal{H}]dP &= \lim_n \sum_i c_i^{(n)} \int_H \chi_{H_i^{(n)}} E[X|\mathcal{H}]dP \\ &= \lim_n \sum_i c_i^{(n)} \int_{H\cap H_i^{(n)}} E[X|\mathcal{H}]dP \\ &= \lim_n \sum_i c_i^{(n)} \int_{H\cap H_i^{(n)}} XdP = \int_H XY dP, \end{aligned} \tag{17}$$

where we replaced the essentially bounded function Y by a bounded function under the integral (by correcting it on a set of P-measure zero) and approximated it pointwise from below by step functions of sets from $\mathcal{H}$. (This is possible, since Y is $\mathcal{H}$-measurable). Then we applied the dominated convergence and the defining property of conditional expectations. By comparing (16) and (17) we conclude by the P-a.s. uniqueness.

Problem 4. Let $f \in \mathcal{V}(0,T)$ be such that

$$\lim_{s\to t} E\big[|f(s,\,\cdot\,) - f(t,\,\cdot\,)|^2\big] = 0 \tag{18}$$

for $s, t \in [0,T]$. Show that the following limit exists in $L^2(P)$

$$\int_0^T f(t,\,\cdot\,)dB_t = \lim_{n\to\infty} \sum_j f(t_j,\,\cdot\,)(\Delta B_j). \tag{19}$$

Check that $f(t,\omega) = g(B_t(\omega))$ for any bounded, continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfies (18).

Remark 0.3. Since $f \in \mathcal{V}(0,T)$ we know already that the l.h.s. of (19) exists as an L^2-limit of elementary functions. But we do not know yet if the particular choice on the r.h.s. of (19) does the job.

Note that for $f(t,\omega) = B_t(\omega)^2$ we verified (19) already in HS8, Problem 5.

Solution. We set $f_n(t,\,\cdot\,) = \sum_j f(t_j,\,\cdot\,)\chi_{[t_j,t_{j+1})}(t)$

$$\begin{aligned}
E\bigg[\bigg|\int_0^T [f(t,\,\cdot\,) - f_n(t,\,\cdot\,)]dB_t\bigg|^2\bigg] &= \int_0^T E\big[|f(t,\,\cdot\,) - f_n(t,\,\cdot\,)|^2\big]dt \\
&= \sum_j \int_{t_j}^{t_{j+1}} E\big[|f(t,\,\cdot\,) - f(t_j,\,\cdot\,)|^2\big]dt \\
&\le \sum_j \sup_{u,v\in[t_j,t_{j+1}]} E\big[|f(u,\,\cdot\,) - f(v,\,\cdot\,)|^2\big](t_{j+1} - t_j) \\
&\le T \sup_{u,v\in[0,T],|u-v|\le 2^{-n}} E\big[|f(u,\,\cdot\,) - f(v,\,\cdot\,)|^2\big] \underset{n\to\infty}{\to} 0,
\end{aligned}$$

since continuous function (valued in any metric space) on a compact set is uniformly continuous. In our case we have a function $[0,T] \ni u \mapsto f(u,\,\cdot\,) \in L^2(P)$ and, by assumption, we have continuity in the respective metric topologies.

Regarding the last question, by continuity of $t \mapsto g(B_t)$, dominated convergence gives

$$\lim_{s\to t} E\big[|g(B_s) - g(B_t)|^2\big] = 0, \tag{20}$$

which verifies (18).

To be discussed in class: 16.01.2026