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Stochastic Differential Equations
Homework Sheet 11

Problem 1. Recall the following definition:

Definition 0.1. Let $\mathcal{V}_{\mathcal{H}}(S,T)$ be the class of functions

$$(t,\omega) \mapsto f(t,\omega) : \mathbb{R}_+ \times \Omega \to \mathbb{R} \tag{1}$$

such that

(i) f is $\mathcal{B}(\mathbb{R}_+) \times \mathcal{F}$-measurable.

(ii') There exists a filtration $\{\mathcal{H}_t\}_{t\in\mathbb{R}_+}$ s.t. $(B_t, \mathcal{H}_t)_{t\in\mathbb{R}_+}$ is a martingale and f is $\{\mathcal{H}_t\}_{t\in\mathbb{R}_+}$-adapted.

(iii) $E\big[\int_S^T f(t,\,\cdot\,)^2 dt\big] < \infty$.

In the context of this definition a function $\varphi \in \mathcal{V}_{\mathcal{H}}(S,T)$ is called elementary if it has the form

$$\varphi(t,\omega) = \sum_{j\geq 0} e_j(\omega)\chi_{[t_j,t_{j+1})}(t), \tag{2}$$

where e_j is $\mathcal{H}_{t_j}$-measurable. Prove the following generalization of the Itô lemma:

Lemma 0.2. If $\varphi \in \mathcal{V}_{\mathcal{H}}(S,T)$ is elementary, then

$$E\bigg[\bigg(\int_S^T \varphi(t,\,\cdot\,)dB_t(\,\cdot\,)\bigg)^2\bigg] = E\bigg[\int_S^T \varphi(t,\,\cdot\,)^2 dt\bigg]. \tag{3}$$

Hint: You can use without proof, that for any filtration $\{\mathcal{H}_t\}_{t\in\mathbb{R}_+}$, for which $(B_t, \mathcal{H}_t)_{t\in\mathbb{R}_+}$ is a continuous martingale, $(B_t^2 - t, \mathcal{H}_t)_{t\in\mathbb{R}_+}$ is also a martingale (cf. Baldi "Stochastic calculus", Theorem 5.16, Example 5.4).

Problem 2. Let $(\Omega, \{\mathcal{H}_t\}_{t\in\mathcal{T}}, \mathcal{F}, P)$ be a filtered probability space. Let $\tau : \Omega \to \overline{\mathcal{T}}$ be a stopping time, i.e. $\{\tau \leq t\} \subset \mathcal{H}_t$ for any $t \in \mathcal{T}$.

(a) Show that for any $s \in \mathcal{T}$ also $\tau(\,\cdot\,) \wedge s := \min(\tau(\,\cdot\,), s)$ is a stopping time.

(b) Show that the stopping time $\tau \wedge s$ is bounded by s. That is, $P(\tau \wedge s \leq s) = 1$.

Problem 3. Let $X \in L^\infty(\Omega, \mathcal{F}, P)$ and $Y \in L^1(\Omega, \mathcal{F}, P)$. Show that

$$E[X \cdot Y|\mathcal{H}] = X \cdot E[Y|\mathcal{H}], \tag{4}$$

if X is $\mathcal{H}$-measurable. (Recall that in the lecture we covered $X \in L^\infty(\Omega, \mathcal{F}, P), Y \in L^2(\Omega, \mathcal{F}, P)$ and $X, Y \in L^2(\Omega, \mathcal{F}, P)$.)

Problem 4. Let $f \in \mathcal{V}(0, T)$ be such that

$$\lim_{s \to t} E\big[|f(s, \cdot) - f(t, \cdot)|^2\big] = 0 \tag{5}$$

for $s, t \in [0, T]$. Show that the following limit exists in $L^2(P)$

$$\int_0^T f(t, \cdot) dB_t = \lim_{n \to \infty} \sum_j f(t_j, \cdot)(\Delta B_j). \tag{6}$$

Check that $f(t, \omega) = g(B_t(\omega))$ for any bounded, continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfies (5).

Remark 0.3. Since $f \in \mathcal{V}(0, T)$ we know already that the l.h.s. of (6) exists as an L^2-limit of elementary functions. But we do not know yet if the particular choice on the r.h.s. of (6) does the job.

Note that for $f(t, \omega) = B_t(\omega)^2$ we have verified (6) already in HS8, Problem 5.

To be discussed in class: 16.01.2026