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Stochastic Differential Equations
Homework Sheet 12 - solutions

Problem 1. In this and the following exercises $\{B_t\}_{t\in\mathbb{R}_+}$ is the one-dimensional Brownian motion with $B_0 = 0$, unless stated otherwise. Use the Itô formula to find representations of the expressions

$$X_t = \int_0^t \exp(B_s) dB_s, \tag{1}$$

$$Y_t = \int_0^t B_s \exp(B_s^2) dB_s, \tag{2}$$

$$Z_t = \int_0^t \exp(s) dB_s, \tag{3}$$

which do not contain stochastic integrals.

Solution. Regarding (1), consider an analogous expression for $s \mapsto \tilde{B}_s \in C^1(\mathbb{R})$:

$$\tilde{X}_t = \int_0^t \exp(\tilde{B}_s) d\tilde{B}_s = \exp(\tilde{B}_t) - 1. \tag{4}$$

Accordingly, let us apply the Itô formula to $g(x) = \exp(x) - 1$. We get

$$dg(B_t) = g'(B_t)dB_t + \frac{1}{2}g''(B_t)dt = \exp(B_t)dB_t + \frac{1}{2}\exp(B_t)dt. \tag{5}$$

Hence,

$$(\exp(B_t) - 1) - \underbrace{(\exp(B_0) - 1)}_{=0} = \int_0^t \exp(B_s)dB_s + \frac{1}{2}\int_0^t \exp(B_s)ds, \tag{6}$$

$$\int_0^t \exp(B_s)dB_s = (\exp(B_t) - 1) - \frac{1}{2}\int_0^t \exp(B_s)ds. \tag{7}$$

Let us now consider (2). Analogously as above, we write

$$\tilde{Y}_t = \int_0^t \tilde{B}_s \exp(\tilde{B}_s^2)d\tilde{B}_s = \frac{1}{2}\int_0^t \exp(\tilde{B}_s^2)d(\tilde{B}_s)^2 = \frac{1}{2}\big(\exp(\tilde{B}_t^2) - 1\big). \tag{8}$$

We set $g(x) = \exp(x^2) - 1$, then

$$dg(B_t) = g'(B_t)dB_t + \frac{1}{2}g''(B_t)dt = 2\exp(B_t^2)B_t dB_t + \frac{1}{2}\exp(B_t^2)[2 + (2B_t)^2]dt. \tag{9}$$

Consequently,

$$(\exp(B_t^2)-1)-\underbrace{(\exp(B_0^2)-1)}_{=0}=2\int_0^t \exp(B_s^2)B_s dB_s+\frac{1}{2}\int_0^t \exp(B_s^2)[2+(2B_s)^2]ds \tag{10}$$

Regarding (3), we use the integration by parts formula to write

$$Z_t=\int_0^t \exp(s)dB_s=\exp(t)B_t-\int_0^t \exp(s)B_s ds. \tag{11}$$

Problem 2. Use the Itô formula to write the processes

$$Y_t=2+t+\exp B_t, \tag{12}$$

$$Y_t=\mathrm{e}^{\frac{1}{2}t}\cos(B_t), \tag{13}$$

$$Y_t=(B_t+t)\exp(-B_t-\frac{1}{2}t) \tag{14}$$

in the standard form. That is, find $\tilde{u}$, $\tilde{v}$ s.t.

$$dY_t=\tilde{u}(t,\cdot)dt+\tilde{v}(t,\cdot)dB_t. \tag{15}$$

Which of the processes are martingales w.r.t. the natural filtration of the Brownian motion?

Solution. We choose $dX_t=udt+vdB_t=dB_t$, i.e., $u\equiv 0$, $v\equiv 1$ in all examples. Recall the Itô formula

$$dY_t=\frac{\partial g}{\partial t}(t,X_t)dt+\frac{\partial g}{\partial x}(t,X_t)dX_t+\frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X_t)(dX_t)^2, \tag{16}$$

with $dt\cdot dt=dt\cdot dB_t=dB_t\cdot dt=0, \quad dB_t\cdot dB_t=dt.$

- Regarding (12),

$$dY_t=dt+\exp(B_t)dB_t+\frac{1}{2}\exp(B_t)dt=\big(1+\frac{1}{2}\exp(B_t)\big)dt+\exp(B_t)dB_t. \tag{17}$$

  Thus $\tilde{u}(t,\omega):=\big(1+\frac{1}{2}\exp(B_t(\omega))\big)$, $\tilde{v}(t,\omega):=\exp(B_t(\omega))$.

- Regarding (13),

$$dY_t=\frac{1}{2}\mathrm{e}^{\frac{1}{2}t}\cos(B_t)dt-\mathrm{e}^{\frac{1}{2}t}\sin(B_t)dB_t-\frac{1}{2}\mathrm{e}^{\frac{1}{2}t}\cos(B_t)dt=-\mathrm{e}^{\frac{1}{2}t}\sin(B_t)dB_t. \tag{18}$$

- Regarding (14),

$$\begin{aligned}dY_t=&(1-\frac{B_t+t}{2})\exp\big(-B_t-\frac{1}{2}t\big)dt+(1-(B_t+t))\exp\big(-B_t-\frac{1}{2}t\big)dB_t\\&-(1-\frac{B_t+t}{2})\exp\big(-B_t-\frac{1}{2}t\big)dt\\=&(1-(B_t+t))\exp\big(-B_t-\frac{1}{2}t\big)dB_t.\end{aligned} \tag{19}$$

Processes (13) and (14) are martingales, since they have the form $dY_t = \tilde{v}(t,\,\cdot\,)dB_t$, i.e., $Y_t = Y_0 + \int_0^t \tilde{v}(s,\,\cdot\,)dB_s$, where Y_0 (deterministic constant) and $\int_0^t \tilde{v}(s,\,\cdot\,)dB_s$ (Itô integral) are martingales.

We still have to show that $Y_t = 2 + t + \exp B_t$ is not a martingale. For this it suffices to show that $Z_t = t + \exp B_t$ is not a martingale, since the constant 2 is a martingale. We compute for $s \geq t$

$$E[Z_s|\mathcal{F}_t] = s + E[\exp B_s|\mathcal{F}_t] = s + \exp(B_t)E[\exp{(B_s - B_t)}]. \tag{20}$$

where we used that $\exp B_t$ is $\mathcal{F}_t$-measurable and $B_s - B_t$ is independent of $\mathcal{F}_t$. Now, since $(B_s - B_t) \sim N(0, |s-t|)$, we have

$$E[Z_s|\mathcal{F}_t] = s + \exp(B_t)\int \exp{(x)}\frac{1}{\sqrt{2\pi|s-t|}}\mathrm{e}^{-\frac{x^2}{2|t-s|}}dx = s + \exp(B_t)\exp(|s-t|/2), \tag{21}$$

where in the last step we used Gaussian integration. If Z_t was a martingale, we would have

$$t + \exp B_t = s + \exp(B_t)\exp(|s-t|/2) \tag{22}$$

for all $s \geq t$. This is clearly impossible, since we can keep t fixed and make s arbitrarily large.

Problem 3. Let $B_t = (B_t^{(1)}, \dots, B_t^{(d)})$ denote the d-dimensional Brownian motion. Following Øksendal, Subsection 4.2, consider the process:

$$dX_t = u_t dt + v_t dB_t, \tag{23}$$

where

$$dX_t = \begin{bmatrix} dX_t^{(1)} \\ \vdots \\ dX_t^{(n)} \end{bmatrix}, \quad u_t = \begin{bmatrix} u_t^{(1)} \\ \vdots \\ u_t^{(n)} \end{bmatrix}, \quad v_t = \begin{bmatrix} v_t^{(1,1)}, & \dots, & v_t^{(1,d)} \\ \vdots & & \vdots \\ v_t^{(n,1)}, & \dots, & v_t^{(n,d)} \end{bmatrix}, \quad dB_t = \begin{bmatrix} dB_t^{(1)} \\ \vdots \\ dB_t^{(d)} \end{bmatrix}. \tag{24}$$

Let g be a C^2 map from $\mathbb{R}_+ \times \mathbb{R}^d$ to $\mathbb{R}^p$:

$$g(t,x) = \begin{bmatrix} g^{(1)}(t,x) \\ \vdots \\ g^{(p)}(t,x) \end{bmatrix}. \tag{25}$$

Then the process $Y_t = g(t, X_t)$ is given by

$$dY_t^{(k)} = \frac{\partial g^{(k)}}{\partial t}(t, X_t)dt + \sum_i \frac{\partial g^{(k)}}{\partial x^{(i)}}(t, X_t)dX_t^{(i)} + \frac{1}{2}\sum_{i,j}\frac{\partial^2 g^{(k)}}{\partial x^{(i)}\partial x^{(j)}}(t, X)dX_t^{(i)}dX_t^{(j)}, \tag{26}$$

where $k \in \{1, \dots, p\}$, $i, j \in \{1, \dots, d\}$ and $dB_t^{(i)}dB_t^{(j)} = \delta_{i,j}dt$, $dB_t^{(i)}dt = dtdB_t^{(i)} = 0$.

Using (26), compute dY_t for the following processes:

$$Y_t = (B_t^{(1)})^2 + (B_t^{(2)})^2 \text{ for } d = 2, \tag{27}$$

$$Y_t = \begin{bmatrix} B_t^{(1)} + B_t^{(2)} + B_t^{(3)} \\ (B_t^{(2)})^2 - B_t^{(1)} B_t^{(3)} \end{bmatrix} \text{ for } d = 3, \tag{28}$$

$$Y_t = |B_t| = \left((B_t^{(1)})^2 + \cdots + (B_t^{(d)})^2\right)^{1/2} \text{ for } d \geq 2, \tag{29}$$

$$Y_t = \exp(v \cdot B_t - \frac{1}{2}|v|^2 t) \text{ for } d \geq 1, \tag{30}$$

where in (30) $v \in \mathbb{R}^d$, $v \cdot B_t := \sum_{i=1}^d v^{(i)} B_t^{(i)}$ and $|v|^2 := (v^{(1)})^2 + \cdots + (v^{(d)})^2$.

Express the result in the form $dY_t = \tilde{u}_t dt + \tilde{v}_t dB_t$ analogous to (23). Identify the vectors $\tilde{u}_t$ and matrices $\tilde{v}_t$.

Remark: In the case of (29), $g(x) = |x|$ is not C^2 at the origin. But $\{B_t\}_{t \in \mathbb{R}_+}$ never hits the origin a.s. when $d \geq 2$, so the Itô formula can be applied anyway.

Solution. In all cases we take $X_t = B_t$. Formula (26) gives:

- Regarding (27),

$$\begin{aligned} dY_t &= 2B_t^{(1)} dB_t^{(1)} + 2B_t^{(2)} dB_t^{(2)} + (dB_t^{(1)})^2 + (dB_t^{(2)})^2 \\ &= 2B_t^{(1)} dB_t^{(1)} + 2B_t^{(2)} dB_t^{(2)} + 2dt \\ &= 2dt + \left[2B_t^{(1)}, 2B_t^{(2)}\right] \begin{bmatrix} dB_t^{(1)} \\ dB_t^{(2)} \end{bmatrix}. \end{aligned} \tag{31}$$

- Regarding (28),

$$\begin{aligned} dY_t &= \begin{bmatrix} dB_t^{(1)} + dB_t^{(2)} + dB_t^{(3)} \\ -B_t^{(3)} dB_t^{(1)} + 2B_t^{(2)} dB_t^{(2)} - B_t^{(1)} dB_t^{(3)} + (dB_t^{(2)})^2 + dB_t^{(1)} dB_t^{(3)} \end{bmatrix} \\ &= \begin{bmatrix} dB_t^{(1)} + dB_t^{(2)} + dB_t^{(3)} \\ -B_t^{(3)} dB_t^{(1)} + 2B_t^{(2)} dB_t^{(2)} - B_t^{(1)} dB_t^{(3)} + dt \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} dt + \begin{bmatrix} 1 & 1 & 1 \\ -B_t^{(3)}, & 2B_t^{(2)}, & -B_t^{(1)} \end{bmatrix} \begin{bmatrix} dB_t^{(1)} \\ dB_t^{(2)} \\ dB_t^{(3)} \end{bmatrix}. \end{aligned} \tag{32}$$

- Regarding (29), we have $g(x) = |x| = \left((x^{(1)})^2 + \cdots + (x^{(d)})^2\right)^{1/2}$ and

$$\frac{\partial g(x)}{\partial x^{(i)}} = \frac{x^{(i)}}{|x|}, \quad \frac{\partial^2 g(x)}{\partial x^{(i)} \partial x^{(j)}} = \frac{\delta_{i,j}}{|x|} - \frac{x^{(i)} x^{(j)}}{|x|^3}. \tag{33}$$

Consequently

$$\begin{aligned}
dY_t &= \sum_i \frac{\partial g^{(k)}}{\partial x^{(i)}}(t, X_t) dX_t^{(i)} + \frac{1}{2}\sum_{i,j} \frac{\partial^2 g^{(k)}}{\partial x^{(i)} \partial x^{(j)}}(t, X) dX_t^{(i)} dX_t^{(j)} \\
&= \sum_i \frac{B_t^{(i)}}{|B_t|} dB_t^{(i)} + \frac{1}{2}\sum_{i,j} \left(\frac{\delta_{i,j}}{|B_t|} - \frac{B_t^{(i)} B_t^{(j)}}{|B_t|^3} \right) dB_t^{(i)} dB_t^{(j)} \\
&= \sum_i \frac{B_t^{(i)}}{|B_t|} dB_t^{(i)} + \frac{1}{2}\sum_{i,j} \left(\frac{\delta_{i,j}}{|B_t|} - \frac{B_t^{(i)} B_t^{(j)} \delta_{i,j}}{|B_t|^3} \right) dt \\
&= \sum_i \frac{B_t^{(i)}}{|B_t|} dB_t^{(i)} + \frac{1}{2}\frac{(d-1)}{|B_t|} dt \\
&= \frac{1}{2}\frac{(d-1)}{|B_t|} dt + \frac{1}{|B_t|} \left[B_t^{(1)}, \dots, B_t^{(d)} \right] \begin{bmatrix} dB_t^{(1)} \\ \vdots \\ dB_t^{(d)} \end{bmatrix}.
\end{aligned} \tag{34}$$

- Regarding (30), we have $g(t, x) = \exp(v \cdot x - \frac{1}{2}|v|^2 t)$ and

$$\frac{\partial g}{\partial t} = -\frac{1}{2}|v|^2 g, \quad \frac{\partial g}{\partial x^{(i)}} = v^{(i)} g, \quad \frac{\partial^2 g}{\partial x^{(i)} \partial x^{(j)}} = v^{(i)} v^{(j)} g. \tag{35}$$

Consequently,

$$\begin{aligned}
dY_t &= \frac{\partial g}{\partial t}(t, X_t) dt + \sum_i \frac{\partial g}{\partial x^{(i)}}(t, X_t) dX_t^{(i)} + \frac{1}{2}\sum_{i,j} \frac{\partial^2 g}{\partial x^{(i)} \partial x^{(j)}}(t, X) dX_t^{(i)} dX_t^{(j)} \\
&= Y_t \left(-\frac{1}{2}|v|^2 dt + \sum_i v^{(i)} dB_t^{(i)} + \frac{1}{2}\sum_{i,j} v^{(i)} v^{(j)} dB_t^{(i)} dB_t^{(j)} \right) \\
&= Y_t \left(-\frac{1}{2}|v|^2 dt + \sum_i v^{(i)} dB_t^{(i)} + \frac{1}{2}\sum_{i,j} v^{(i)} v^{(j)} \delta_{i,j} dt \right) \\
&= Y_t \sum_i v^{(i)} dB_t^{(i)} = Y_t v \cdot dB_t = Y_t \left[v^{(1)}, \dots, v^{(d)} \right] \begin{bmatrix} dB_t^{(1)} \\ \vdots \\ dB_t^{(d)} \end{bmatrix}.
\end{aligned} \tag{36}$$

To be discussed in class: 23.01.2026