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Stochastic Differential Equations
Homework Sheet 12

Problem 1. In this and the following exercises $\{B_t\}_{t\in\mathbb{R}_+}$ is the one-dimensional Brownian motion with $B_0 = 0$, unless stated otherwise. Use the Itô formula to find representations of the expressions

$$X_t = \int_0^t \exp(B_s) dB_s, \tag{1}$$

$$Y_t = \int_0^t B_s \exp(B_s^2) dB_s, \tag{2}$$

$$Z_t = \int_0^t \exp(s) dB_s, \tag{3}$$

which do not contain stochastic integrals.

Problem 2. Use the Itô formula to write the processes

$$Y_t = 2 + t + \exp B_t, \tag{4}$$

$$Y_t = \mathrm{e}^{\frac{1}{2}t} \cos(B_t), \tag{5}$$

$$Y_t = (B_t + t) \exp(-B_t - \frac{1}{2}t) \tag{6}$$

in the standard form. That is, find $\tilde{u}$, $\tilde{v}$ s.t.

$$dY_t = \tilde{u}(t, \cdot)dt + \tilde{v}(t, \cdot)dB_t. \tag{7}$$

Which of the processes are martingales w.r.t. the natural filtration of the Brownian motion?

Problem 3. Let $B_t = (B_t^{(1)}, \dots, B_t^{(d)})$ denote the d-dimensional Brownian motion. Following Øksendal, Subsection 4.2, consider the process:

$$dX_t = u_t dt + v_t dB_t, \tag{8}$$

where

$$dX_t = \begin{bmatrix} dX_t^{(1)} \\ \vdots \\ dX_t^{(n)} \end{bmatrix}, \quad u_t = \begin{bmatrix} u_t^{(1)} \\ \vdots \\ u_t^{(n)} \end{bmatrix}, \quad v_t = \begin{bmatrix} v_t^{(1,1)}, & \dots, & v_t^{(1,d)} \\ \vdots & & \vdots \\ v_t^{(n,1)}, & \dots, & v_t^{(n,d)} \end{bmatrix}, \quad dB_t = \begin{bmatrix} dB_t^{(1)} \\ \vdots \\ dB_t^{(d)} \end{bmatrix}. \tag{9}$$

Let g be a C^2 map from $\mathbb{R}_+ \times \mathbb{R}^d$ to $\mathbb{R}^p$:

$$g(t,x) = \begin{bmatrix} g^{(1)}(t,x) \\ \vdots \\ g^{(p)}(t,x) \end{bmatrix}. \tag{10}$$

Then the process $Y_t = g(t, X_t)$ is given by

$$dY_t^{(k)} = \frac{\partial g^{(k)}}{\partial t}(t,X_t)dt + \sum_i \frac{\partial g^{(k)}}{\partial x^{(i)}}(t,X_t)dX_t^{(i)} + \frac{1}{2}\sum_{i,j}\frac{\partial^2 g^{(k)}}{\partial x^{(i)}\partial x^{(j)}}(t,X)dX_t^{(i)}dX_t^{(j)}, \tag{11}$$

where $k \in \{1,\dots,p\}$, $i,j \in \{1,\dots,d\}$ and $dB_t^{(i)}dB_t^{(j)} = \delta_{i,j}dt$, $dB_t^{(i)}dt = dtdB_t^{(i)} = 0$.

Using (11), compute dY_t for the following processes:

$$Y_t = (B_t^{(1)})^2 + (B_t^{(2)})^2 \text{ for } d = 2, \tag{12}$$

$$Y_t = \begin{bmatrix} B_t^{(1)} + B_t^{(2)} + B_t^{(3)} \\ (B_t^{(2)})^2 - B_t^{(1)}B_t^{(3)} \end{bmatrix} \text{ for } d = 3, \tag{13}$$

$$Y_t = |B_t| = \left((B_t^{(1)})^2 + \cdots + (B_t^{(d)})^2\right)^{1/2} \text{ for } d \geq 2, \tag{14}$$

$$Y_t = \exp(v \cdot B_t - \frac{1}{2}|v|^2 t) \text{ for } d \geq 1, \tag{15}$$

where in (15) $v \in \mathbb{R}^d$, $v \cdot B_t := \sum_{i=1}^d v^{(i)}B_t^{(i)}$ and $|v|^2 := (v^{(1)})^2 + \cdots + (v^{(d)})^2$.

Express the result in the form $dY_t = \tilde{u}_t dt + \tilde{v}_t dB_t$ analogous to (8). Identify the vectors $\tilde{u}_t$ and matrices $\tilde{v}_t$.

Remark: In the case of (14), $g(x) = |x|$ is not C^2 at the origin. But $\{B_t\}_{t\in\mathbb{R}_+}$ never hits the origin a.s. when $d \geq 2$, so the Itô formula can be applied anyway.

To be discussed in class: 23.01.2026