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Stochastic Differential Equations
Homework Sheet 13

Problem 1. Let $\{B_t\}_{t\in\mathbb{R}_+}$ be one-dimensional Brownian motion. Show that the process $Y_t = \mathrm{e}^{B_t}$ satisfies

$$dY_t = \frac{1}{2}Y_t dt + Y_t dB_t. \tag{1}$$

Solution. Recall the Itô formula in the simplest form:

$$dg(B_t) = g'(B_t)dB_t + \frac{1}{2}g''(B_t)dt. \tag{2}$$

We compute

$$dY_t = d(\mathrm{e}^{B_t}) = \mathrm{e}^{B_t}dB_t + \frac{1}{2}\mathrm{e}^{B_t}dt = \frac{1}{2}Y_t dt + Y_t dB_t. \tag{3}$$

Problem 2. For fixed $a, b \in \mathbb{R}$ consider the following 1-dimensional equation

$$dY_t = \frac{b - Y_t}{1-t}dt + dB_t, \quad 0 \le t < 1, Y_0 = a. \tag{4}$$

Verify that

$$Y_t = a(1-t) + bt + (1-t)\int_0^t \frac{dB_s}{1-s}, \quad 0 \le t < 1 \tag{5}$$

solves the equation and prove that $\lim_{t\to 1} Y_t = b$ a.s.

Hint 1: Use the 'more general case' of the Itô formula from the lecture.

Hint 2: Integration by parts may be useful for proving $\lim_{t\to 1} Y_t = b$.

Remark: The process $\{Y_t\}_{t\in[0,1)}$ is called the Brownian bridge.

Solution. Recall the Itô formula

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)(dX_t)^2, \tag{6}$$

with $dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0, \quad dB_t \cdot dB_t = dt$. Choose

$$X_t = \int_0^t \frac{dB_s}{1-s}, \tag{7}$$

that is, equivalently,

$$dX_t = \frac{1}{1-t} dB_t. \tag{8}$$

Then we have, by (9)

$$Y_t = a(1-t) + bt + (1-t)\int_0^t \frac{dB_s}{1-s} = a(1-t) + bt + (1-t)X_t =: g(t, X_t). \tag{9}$$

By the Itô formula, we have

$$\begin{aligned} dY_t &= (-a+b-X_t)dt + (1-t)dX_t = (-a+b-X_t)dt + dB_t \\ &= \frac{b-Y_t}{1-t}dt + dB_t, \end{aligned} \tag{10}$$

where in the second step we used (8) and in the third step (9).

Let us now prove that $\lim_{t\to 1} Y_t = b$. It is clear from (9), that it suffices to show that

$$\lim_{t\to 1}(1-t)\int_0^t \frac{dB_s}{1-s} = 0. \tag{11}$$

For this purpose, we apply integration by parts

$$(1-t)\int_0^t \frac{dB_s}{1-s} = B_t - (1-t)\int_0^t B_s d(\frac{1}{1-s}). \tag{12}$$

We choose a new variable $u := \frac{1-t}{1-s}$ in the integral. Then

$$(12) = B_t - \int_{1-t}^1 B_{1-\frac{1}{u}(1-t)} du. \tag{13}$$

B_s above and in the following stand for $B_s(\omega)$ for some fixed ω, since we want to obtain the limit almost surely. Almost surely means here, that we restrict attention to ω for which $s \mapsto B_s(\omega)$ is continuous. Such ω form a subset of Ω of measure one. Furthermore, we extend $s \mapsto B_s(\omega)$ to negative s by zero and denote the resulting function by $\mathbb{R} \ni s \mapsto \tilde{B}_s(\omega)$. (It is continuous since $B_0 = 0$). Given all this, we can write

$$B_t - \int_{1-t}^1 B_{1-\frac{1}{u}(1-t)} du = B_t - \int_0^1 \tilde{B}_{1-\frac{1}{u}(1-t)} du, \tag{14}$$

since the integrand vanishes for $u \in [0, 1-t]$ as it corresponds to negative s. Now we get

$$\lim_{t\to 1}\int_0^1 \tilde{B}_{1-\frac{1}{u}(1-t)} du = B_1 \tag{15}$$

by continuity of $\mathbb{R} \ni s \mapsto \tilde{B}_s(\omega)$ and dominated convergence.

Problem 3. Show that there is a solution $\{X_t\}_{t\in\mathbb{R}_+}$ of the one-dimensional stochastic differential equation

$$dX_t = \ln(1+X_t^2)dt + X_t dB_t, \quad X_0 = a \in \mathbb{R}. \tag{16}$$

Does the result still hold if we replace $X_t dB_t$ with $\chi_{\{X_t>0\}} X_t dB_t$ above?

Hint: Verify the assumptions of the 'existence and uniqueness theorem' from the lecture.

Solution. We recall the theorem in a slightly shortened form:

Theorem 0.1. *Let $(\Omega, \mathcal{F}, P)$ be the probability space of the d-dimensional Brownian motion. Let $T > 0$ and $b(\,\cdot\,,\,\cdot\,) : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$, $\sigma(\,\cdot\,,\,\cdot\,) : [0,T]\times\mathbb{R}^n \to \mathbb{R}^{n\times d}$ be measurable functions satisfying*

$$|b(t,x)| + |\sigma(t,x)| \le C(1+|x|), \quad x\in\mathbb{R}^n, t\in[0,T] \tag{17}$$

for some constant C and s.t. the Lipschitz property property holds

$$|b(t,x)-b(t,y)| + |\sigma(t,x)-\sigma(t,y)| \le D|x-y|, \quad x,y\in\mathbb{R}^d, t\in[0,T] \tag{18}$$

for some constant D. Let Z be a random variable which is independent of the σ-algebra $\mathcal{F}_\infty^{(d)}$ generated by $B_s(\,\cdot\,)$, $s\ge 0$, and s.t. $E[|Z|^2] < \infty$. Then the stochastic differential equation

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t, \quad t\in[0,T], \quad X_0 = Z, \tag{19}$$

has a unique t-continuous solution $\{X_t\}_{t\in[0,T]}$.

We have $b(t,x) = \ln(1+x^2)$ and $\sigma(t,x) = x$. As measurability is obvious, it suffices to check (17) and (18). Regarding (17), we first note that $1+u \le \mathrm{e}^u$ (obvious by drawing graphs) hence, for $u>0$,

$$\ln(u) \le \ln(1+u) \le u. \tag{20}$$

Thus, for $u = (1+|x|^2)^{1/2}$,

$$\ln(1+|x|^2) \le 2(1+|x|^2)^{1/2} \le 2(1+|x|). \tag{21}$$

Consequently,

$$\frac{|b(t,x)|+|\sigma(t,x)|}{(1+|x|)} \le \frac{\ln(1+x^2)+|x|}{(1+|x|)} \le \frac{2+3|x|}{1+|x|} \le 3. \tag{22}$$

Regarding (18), obviously,

$$|\sigma(t,x)-\sigma(t,y)| \le |x-y|. \tag{23}$$

Furthermore, setting $z := x - y$, we have

$$|b(t,x) - b(t,y)| = |\ln(1+(y+z)^2) - \ln(1+y^2)| = \Big|\int_0^1 \frac{d}{du}\ln(1+(y+uz)^2)\,du\Big|$$
$$= \Big|\int_0^1 \frac{2(y+uz)z}{1+(y+uz)^2}\,du\Big| \le 2\sqrt{2}|z|. \tag{24}$$

Here in the last step we used the Cauchy-Schwarz inequality for series as follows:

$$|y+uz| \le 1 + |y+uz| = 1\cdot 1 + 1\cdot|y+uz| \le (1^2+1^2)^{1/2}(1^2+|y+uz|^2)^{1/2}. \tag{25}$$

Regarding the case of $\sigma(t,x) = \chi_{\{x>0\}}x$ the only part of the proof that has to be reconsidered is (23). Clearly, for $x > 0$ and $y > 0$ (23) holds as before. For $x \le 0$, $y \le 0$ (23) holds trivially. It suffices to cover the mixed case $x > 0$, $y \le 0$:

$$|\sigma(t,x) - \sigma(t,y)| = |\chi_{\{x>0\}}x - \chi_{\{y>0\}}y| = |\chi_{\{x>0\}}x| = x \le x - y \le |x-y|, \tag{26}$$

where in the next-to-the-last step we made use of the fact that y is non-positive.

Side remark: Since $x \mapsto \chi_{\{x>0\}}$ is discontinuous, Lipschitz continuity seems to be in danger. However, by drawing a graph, it is clear that $x \mapsto \chi_{\{x>0\}}x$ is continuous. This is what saves the estimate.

Problem 4. Solve the following equation:

$$dY_t = \mu Y_t dt + \sigma dB_t, \quad Y_0 = 0, \tag{27}$$

where μ, σ are real coefficients. The solution is called the *Ornstein-Uhlenbeck process* with $Y_0 = 0$.

Solution. We pick $\tilde{B} \in C^1(\mathbb{R}_+)$ and consider an ordinary differential equation of the same form as (27):

$$\frac{d\tilde{Y}_t}{dt} - \tilde{\mu}\tilde{Y}_t = \tilde{\sigma}\frac{d\tilde{B}_t}{dt}. \tag{28}$$

A strategy to solve such inhomogeneous linear ODE is the following:

(i) First, solve the corresponding homogeneous equation:

$$\frac{d\tilde{Y}_t}{dt} - \tilde{\mu}\tilde{Y}_t = 0. \tag{29}$$

Clearly, this is solved by $\tilde{Y}_t = C e^{\tilde{\mu}t}$.

(ii) Next, we look for a solution of the inhomogeneous equation in the form $\tilde{Y}_t = C_t e^{\tilde{\mu}t}$, where we let the constant C depend on time (this is called the 'method of variation of coefficients'). By substituting $\tilde{Y}_t = C_t e^{\tilde{\mu}t}$ to (28), we get

$$e^{\tilde{\mu}t}\frac{d\tilde{C}_t}{dt} = \tilde{\sigma}\frac{d\tilde{B}_t}{dt} \quad \Rightarrow \quad C_t = \tilde{\sigma}\int_0^t e^{-\tilde{\mu}s}\frac{d\tilde{B}_s}{ds}ds \tag{30}$$

Thus the solution of the ODE (28) has the form

$$\tilde{Y}_t = e^{\tilde{\mu}t}\tilde{\sigma}\int_0^t e^{-\tilde{\mu}s}\frac{d\tilde{B}_s}{ds}ds. \tag{31}$$

Now we come back to the SDE (27). Formula (31) suggests that:

$$Y_t = \mathrm{e}^{\tilde{\mu}t}\tilde{\sigma}\int_0^t \mathrm{e}^{-\tilde{\mu}s} dB_s, \tag{32}$$

where the integral is now interpreted as an Itô integral. We restate the Itô formula for $Y_t = g(t, X_t)$

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)(dX_t)^2, \tag{33}$$

with $dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0, \quad dB_t \cdot dB_t = dt$. Choose

$$X_t = \tilde{\sigma}\int_0^t \mathrm{e}^{-\tilde{\mu}s} dB_s \quad \text{i.e.} \quad dX_t = \tilde{\sigma}\mathrm{e}^{-\tilde{\mu}t} dB_t. \tag{34}$$

$$Y_t = g(t, X_t) := \mathrm{e}^{\tilde{\mu}t} X_t. \tag{35}$$

By (33),

$$dY_t = \tilde{\mu}\mathrm{e}^{\tilde{\mu}t} X_t + \mathrm{e}^{\tilde{\mu}t}\tilde{\sigma}(\mathrm{e}^{-\tilde{\mu}t} dB_t) = \tilde{\mu} Y_t + \tilde{\sigma} dB_t. \tag{36}$$

Thus the equation is satisfied for $\mu = \tilde{\mu}$ and $\sigma = \tilde{\sigma}$.

Side remark: The Ornstein-Uhlenbeck process is the Brownian motion with friction. It corresponds to the Newton equation

$$m\frac{dv}{dt} = -\gamma v + \sigma' \frac{dB_t}{dt} \tag{37}$$

describing velocity v of a fluid particle of mass m moving under a velocity dependent friction force $-\gamma v$ and experiencing random collisions with other particles, modeled by the white noise term $\sigma' \frac{dB_t}{dt}$.

Problem 5. Let $(B^{(1)}, B^{(2)})$ be two-dimensional Brownian motion. We define the complex Brownian motion as follows

$$\mathbb{B}_t = B_t^{(1)} + \mathrm{i}B_t^{(2)}, \tag{38}$$

where i is the imaginary unit. Let $F(z) = F(x^{(1)} + \mathrm{i}x^{(2)}) = u(x^{(1)}, x^{(2)}) + \mathrm{i}v(x^{(1)}, x^{(2)})$ be an analytic function, i.e. F satisfies the Cauchy-Riemann equations

$$\frac{\partial u(x^{(1)}, x^{(2)})}{\partial x^{(1)}} = \frac{\partial v(x^{(1)}, x^{(2)})}{\partial x^{(2)}}, \quad \frac{\partial u(x^{(1)}, x^{(2)})}{\partial x^{(2)}} = -\frac{\partial v(x^{(1)}, x^{(2)})}{\partial x^{(1)}}, \quad z = x^{(1)} + \mathrm{i}x^{(2)}, \tag{39}$$

and we define $Z_t = F(\mathbb{B}_t)$. Prove that

$$dZ_t = F'(\mathbb{B}_t) d\mathbb{B}_t. \tag{40}$$

Use this to solve the complex stochastic differential equation

$$dZ_t = \alpha Z_t d\mathbb{B}_t, \quad Z_0 = 1. \tag{41}$$

Solution. We recall the Itô formula for multi-dimensional Brownian motion from HS12 and Øksendal, Subsection 4.2: Let $B_t = (B_t^{(1)}, B_t^{(2)})$ denote the 2-dimensional Brownian motion. Consider the process:

$$dX_t = u_t dt + v_t dB_t, \tag{42}$$

where

$$dX_t = \begin{bmatrix} dX_t^{(1)} \\ dX_t^{(2)} \end{bmatrix}, \quad u_t = \begin{bmatrix} u_t^{(1)} \\ u_t^{(2)} \end{bmatrix}, \quad v_t = \begin{bmatrix} v_t^{(1,1)}, & v_t^{(1,2)} \\ v_t^{(2,1)}, & v_t^{(2,2)} \end{bmatrix}, \quad dB_t = \begin{bmatrix} dB_t^{(1)} \\ dB_t^{(2)} \end{bmatrix}. \tag{43}$$

Let g be a C^2 map from $\mathbb{R}_+ \times \mathbb{R}^2$ to $\mathbb{R}^2$:

$$g(t,x) = \begin{bmatrix} g^{(1)}(t,x) \\ g^{(2)}(t,x) \end{bmatrix}. \tag{44}$$

Then the process $Y_t = g(t, X_t)$ is given by

$$dY_t^{(k)} = \frac{\partial g^{(k)}}{\partial t}(t, X_t)dt + \sum_i \frac{\partial g^{(k)}}{\partial x^{(i)}}(t, X_t)dX_t^{(i)} + \frac{1}{2}\sum_{i,j} \frac{\partial^2 g^{(k)}}{\partial x^{(i)}\partial x^{(j)}}(t, X)dX_t^{(i)}dX_t^{(j)}, \tag{45}$$

where $k \in \{1,2\}$, $i, j \in \{1,2\}$ and $dB_t^{(i)}dB_t^{(j)} = \delta_{i,j}dt$, $dB_t^{(i)}dt = dtdB_t^{(i)} = 0$.
We denote $x = (x^{(1)}, x^{(2)})$ and $z = x^{(1)} + \mathrm{i}x^{(2)}$. We choose $X_t = B_t$ and

$$g(t,x) = \begin{bmatrix} u(x) \\ v(x) \end{bmatrix}, \tag{46}$$

which is a representation of the complex valued function F as a pair consisting of its real (u) and imaginary (v) part. As this function is t-independent, the first term on the r.h.s. of (45) is zero. The second term has the form,

$$\begin{bmatrix} \sum_i \frac{\partial g^{(1)}}{\partial x^{(i)}}(t, X_t)dX_t^{(i)} \\ \sum_i \frac{\partial g^{(2)}}{\partial x^{(i)}}(t, X_t)dX_t^{(i)} \end{bmatrix} = \begin{bmatrix} \frac{\partial u(x)}{\partial x^{(1)}}, & \frac{\partial u(x)}{\partial x^{(2)}} \\ \frac{\partial v(x)}{\partial x^{(1)}}, & \frac{\partial v(x)}{\partial x^{(2)}} \end{bmatrix} \begin{bmatrix} dB^{(1)} \\ dB^{(2)} \end{bmatrix} = \begin{bmatrix} \frac{\partial u(x)}{\partial x^{(1)}}, & \frac{\partial u(x)}{\partial x^{(2)}} \\ -\frac{\partial u(x)}{\partial x^{(2)}}, & \frac{\partial u(x)}{\partial x^{(1)}} \end{bmatrix} \begin{bmatrix} dB^{(1)} \\ dB^{(2)} \end{bmatrix}, \tag{47}$$

where, in the second step, we made use of the Cauchy-Riemann equations. On the other hand, the r.h.s. of (40) gives

$$\begin{aligned} F'(\mathbb{B}_t)d\mathbb{B}_t &= \frac{1}{2}\left(\frac{\partial}{\partial x^{(1)}} - \mathrm{i}\frac{\partial}{\partial x^{(2)}}\right)(u(x) + \mathrm{i}v(x))(dB_t^{(1)} + \mathrm{i}dB_t^{(2)}) \\ &= \frac{1}{2}\left(\frac{\partial u(x)}{\partial x^{(1)}} + \mathrm{i}\frac{\partial v(x)}{\partial x^{(1)}} - \mathrm{i}\frac{\partial u(x)}{\partial x^{(2)}} + \frac{\partial v(x)}{\partial x^{(2)}}\right)(dB_t^{(1)} + \mathrm{i}dB_t^{(2)}) \\ &= \frac{1}{2}\left(\frac{\partial u(x)}{\partial x^{(1)}} - \mathrm{i}\frac{\partial u(x)}{\partial x^{(2)}} - \mathrm{i}\frac{\partial u(x)}{\partial x^{(2)}} + \frac{\partial u(x)}{\partial x^{(1)}}\right)(dB_t^{(1)} + \mathrm{i}dB_t^{(2)}) \\ &= \left(\frac{\partial u(x)}{\partial x^{(1)}} - \mathrm{i}\frac{\partial u(x)}{\partial x^{(2)}}\right)(dB_t^{(1)} + \mathrm{i}dB_t^{(2)}) \\ &= \frac{\partial u(x)}{\partial x^{(1)}}dB_t^{(1)} + \frac{\partial u(x)}{\partial x^{(2)}}dB_t^{(2)} + \mathrm{i}\left(\frac{\partial u(x)}{\partial x^{(1)}}dB_t^{(2)} - \frac{\partial u(x)}{\partial x^{(2)}}dB_t^{(1)}\right). \end{aligned} \tag{48}$$

Rewriting the real and imaginary parts of this expression as components of a column vector we obtain (47). We still have to show that the last term on the r.h.s. of (45) vanishes. By the Cauchy-Riemann equations:

$$\frac{\partial^2 g^{(1)}}{\partial x^{(1)}\partial x^{(1)}} = \frac{\partial^2 u}{\partial x^{(1)}\partial x^{(1)}} = \frac{\partial^2 v}{\partial x^{(1)}\partial x^{(2)}} = -\frac{\partial^2 u}{\partial x^{(2)}\partial x^{(2)}} \tag{49}$$

Consequently,

$$\begin{aligned}&\sum_{i,j} \frac{\partial^2 g^{(1)}}{\partial x^{(i)}\partial x^{(j)}}(t, X) dX_t^{(i)} dX_t^{(j)} \\ &= \frac{\partial^2 u}{\partial x^{(1)}\partial x^{(1)}}(dB_t^{(1)})^2 + 2\frac{\partial^2 u}{\partial x^{(1)}\partial x^{(2)}} dB_t^{(1)} dB_t^{(2)} + \frac{\partial^2 u}{\partial x^{(2)}\partial x^{(2)}}(dB_t^{(2)})^2 = 0,\end{aligned} \tag{50}$$

where in the last step we used that $(dB^{(1)})^2 = (dB^{(2)})^2 = dt$, $dB^{(1)}dB^{(2)} = 0$ and (49). The case of $g^{(2)}$ is treated analogously.

Regarding the last question, it is clear from (40) that the solution of (41) is $Z_t = \mathrm{e}^{\alpha \mathbb{B}_t}$.

To be discussed in class: 30.01.2026