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AMU Poznań

Winter semester 2025/26

Stochastic Differential Equations
Homework Sheet 13

Problem 1. Let $\{B_t\}_{t\in\mathbb{R}_+}$ be one-dimensional Brownian motion. Show that the process $Y_t = \mathrm{e}^{B_t}$ satisfies

$$dY_t = \frac{1}{2}Y_t dt + Y_t dB_t. \tag{1}$$

Problem 2. For fixed $a, b \in \mathbb{R}$ consider the following 1-dimensional equation

$$dY_t = \frac{b - Y_t}{1-t}dt + dB_t, \quad 0 \leq t < 1, Y_0 = a. \tag{2}$$

Verify that

$$Y_t = a(1-t) + bt + (1-t)\int_0^t \frac{dB_s}{1-s}, \quad 0 \leq t < 1 \tag{3}$$

solves the equation and prove that $\lim_{t\to 1} Y_t = b$ a.s.

Hint 1: Use the 'more general case' of the Itô formula from the lecture.

Hint 2: Integration by parts may be useful for proving $\lim_{t\to 1} Y_t = b$.

Remark: The process $\{Y_t\}_{t\in[0,1)}$ is called the Brownian bridge.

Problem 3. Show that there exists a solution $\{X_t\}_{t\in\mathbb{R}_+}$ of the one-dimensional stochastic differential equation

$$dX_t = \ln(1 + X_t^2)dt + X_t dB_t, \quad X_0 = a \in \mathbb{R}. \tag{4}$$

Does the result still hold if we replace $X_t dB_t$ with $\chi_{\{X_t>0\}}X_t dB_t$ above?

Hint: Verify the assumptions of the 'existence and uniqueness theorem' from the lecture.

Problem 4. Solve the following equation:

$$dY_t = \mu Y_t dt + \sigma dB_t, \quad Y_0 = 0, \tag{5}$$

where μ, σ are real coefficients. The solution is called the *Ornstein-Uhlenbeck process* with $Y_0 = 0$.

Problem 5. Let $(B^{(1)}, B^{(2)})$ be two-dimensional Brownian motion. We define the complex Brownian motion as follows

$$\mathbb{B}_t = B_t^{(1)} + \mathrm{i}B_t^{(2)}, \tag{6}$$

where i is the imaginary unit. Let $F(z) = F(x^{(1)} + \mathrm{i}x^{(2)}) = u(x^{(1)}, x^{(2)}) + \mathrm{i}v(x^{(1)}, x^{(2)})$ be an analytic function, i.e. F satisfies the Cauchy-Riemann equations:

$$\frac{\partial u(x^{(1)}, x^{(2)})}{\partial x^{(1)}} = \frac{\partial v(x^{(1)}, x^{(2)})}{\partial x^{(2)}}, \quad \frac{\partial u(x^{(1)}, x^{(2)})}{\partial x^{(2)}} = -\frac{\partial v(x^{(1)}, x^{(2)})}{\partial x^{(1)}}, \quad z = x^{(1)} + \mathrm{i}x^{(2)}, \tag{7}$$

and we define $Z_t = F(\mathbb{B}_t)$. Prove that

$$dZ_t = F'(\mathbb{B}_t)d\mathbb{B}_t. \tag{8}$$

Use this to solve the complex stochastic differential equation

$$dZ_t = \alpha Z_t d\mathbb{B}_t, \quad Z_0 = 1. \tag{9}$$

To be discussed in class: 30.01.2026