Prof. UAM dr hab. Wojciech Dybalski

AMU Poznań
Winter semester 2025/26

Stochastic Differential Equations
Homework Sheet 2

Problem 1. Let $(\Omega, \mathcal{F}, P) = (\mathbb{R}_+, \mathcal{B}, \mu)$, where $\mathbb{R}_+ = [0, \infty)$, $\mathcal{B}$ denotes the Borel σ-algebra on $[0, \infty)$ and μ is a probability measure on $[0, \infty)$ with no mass on single points, i.e. $\mu(\{x\}) = 0$ for every $x \in \mathbb{R}_+$. Define

$$X_t(\omega) = \begin{cases} 1 & \text{if } t = \omega, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

and

$$Y_t(\omega) = 0 \quad \text{for all} \quad (t, \omega) \in \mathbb{R}_+ \times \mathbb{R}_+. \tag{2}$$

Prove that $\{X_t\}_{t\in\mathbb{R}_+}$ and $\{Y_t\}_{t\in\mathbb{R}_+}$ have the same finite-dimensional distributions and that $\{X_t\}_{t\in\mathbb{R}_+}$ is a version of $\{Y_t\}_{t\in\mathbb{R}_+}$. (Note that, nevertheless, $t \mapsto Y_t(\omega)$ is continuous for all ω and $t \mapsto X_t(\omega)$ is discontinuous for all ω).

Problem 2. Let X and Y be two independent random variables with finite variances. Show that

$$\mathrm{var}(X + Y) = \mathrm{var}(X) + \mathrm{var}(Y).$$

Problem 3. Let X be a real-valued random variable with finite second moment, i.e., $E[X^2] < \infty$.

(a) Let $Y \geq 0$ be any nonnegative random variable and let $a > 0$. Show that

$$P(Y \geq a) \leq \frac{E[Y]}{a}.$$

(b) Apply part (a) with $Y = (X - E[X])^2$ and $a = \varepsilon^2$ for $\varepsilon > 0$. Conclude that

$$P\big(|X - E[X]| \geq \varepsilon\big) \leq \frac{E[(X - E[X])^2]}{\varepsilon^2}.$$

(c) Recall that $\mathrm{var}(X) := E[(X - E[X])^2]$. Rewrite the bound from part (b) in terms of $\mathrm{var}(X)$.

Problem 4. Let $\{B_t\}_{t\in\mathbb{R}_+}$ be 2-dimensional Brownian motion and set

$$D_\rho := \{\, y \in \mathbb{R}^2 \,:\, |y| < \rho\} \quad \text{for} \quad \rho > 0, \tag{3}$$

where $|y| := \sqrt{y_1^2 + y_2^2}$. Compute $P^{x=0}(B_t \in D_\rho)$ for some fixed $t > 0$.

Problem 5. Show that:

(a) $I := \int_{\mathbb{R}} \mathrm{e}^{-\frac{x^2}{2\sigma^2}} dx = \sqrt{2\pi\sigma^2}$.
Hint: For example, write $I^2 = \int_{\mathbb{R}^2} \mathrm{e}^{-\frac{1}{2\sigma^2}(x^2+y^2)} dxdy$ and go to polar coordinates.

(b) $\int_{\mathbb{R}} \mathrm{e}^{ikx} \mathrm{e}^{-\frac{x^2}{2\sigma^2}} dx = \sqrt{2\pi\sigma^2}\, \mathrm{e}^{-\frac{k^2\sigma^2}{2}}$.

(c) $\int_{\mathbb{R}^d} \mathrm{e}^{i\langle k,x\rangle} \mathrm{e}^{-\frac{\langle x, C^{-1}x\rangle}{2}} dx = \sqrt{(2\pi)^d \det(C)}\, \mathrm{e}^{-\frac{\langle k, Ck\rangle}{2}}$, where C is a $d \times d$ symmetric matrix with strictly positive eigenvalues. Hint: Diagonalize the matrix, change variables and use the previous item.

To be discussed in class: 17.10.2025