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Algebraic Quantum Field Theory
Homework Sheet 2

Problem 1. A large class of automorphisms of $\mathcal{W}$ is obtained as follows

$$\alpha(W(z)) = c(z)W(Sz) \tag{1}$$

where $c(z) \in \mathbb{C}\backslash\{0\}$ and $S : \mathbb{C}^n \to \mathbb{C}^n$ a continuous bijection. Show that Weyl relations impose the following restrictions on c, S:

$$c(z+z') = c(z)c(z'), \qquad c(-z) = \overline{c(z)}, \qquad |c(z)| = 1, \tag{2}$$

$$S(z+z') = S(z) + S(z'), \quad S(-z) = -S(z), \quad \mathrm{Im}\langle Sz, Sz'\rangle = \mathrm{Im}\langle z, z'\rangle. \tag{3}$$

Note that these relations imply that $z \mapsto S(z)$ is a real-linear map and that S is symplectic.

Solution: Let us compute

$$\begin{aligned} 0 &= \alpha\big(W(z)W(z') - e^{\frac{i}{2}\mathrm{Im}\langle z|z'\rangle}W(z+z')\big) \\ &= c(z)c(z')W(S(z))W(S(z')) - e^{\frac{i}{2}\mathrm{Im}\langle z|z'\rangle}c(z+z')W(S(z+z')) \\ &= c(z)c(z')e^{\frac{i}{2}\mathrm{Im}\langle S(z)|S(z')\rangle}W(S(z)+S(z')) - e^{\frac{i}{2}\mathrm{Im}\langle z|z'\rangle}c(z+z')W(S(z+z')). \end{aligned} \tag{4}$$

Hence

$$S(z+z') = S(z) + S(z'). \tag{5}$$

and

$$\frac{c(z)c(z')}{c(z+z')} = e^{\frac{i}{2}(\mathrm{Im}\langle z|z'\rangle - \mathrm{Im}\langle S(z)|S(z')\rangle)}. \tag{6}$$

On the other hand

$$\begin{aligned} 0 = \alpha(W(z)^* - W(-z)) &= (\alpha(W(z)))^* - \alpha(W(-z)) \\ &= (c(z)W(S(z)))^* - c(-z)W(S(-z)) \\ &= \overline{c(z)}W(-S(z)) - c(-z)W(S(-z)). \end{aligned} \tag{7}$$

Which gives

$$S(-z) = -S(z), \quad \overline{c(z)} = c(-z). \tag{8}$$

Now from $\alpha(1) = 1$ we get $c(0)W(S(0)) = W(0)$. Hence $c(0) = 1$ and $S(0) = 0$.

Now let us set

$$\frac{1}{2}(\mathrm{Im}\langle z|z'\rangle - \mathrm{Im}\langle S(z)|S(z')\rangle) =: \phi(z,z') \tag{9}$$

Then (6) gives

$$\frac{c(z)c(z')}{c(z+z')} = e^{i\phi(z,z')} \tag{10}$$

Since $S(-z) = -S(z)$, we have $\phi(-z,-z') = \phi(z,z')$. But since $c(-z) = \overline{c(z)}$, we have

$$e^{i\phi(z,z')} = \frac{c(-z)c(-z')}{c(-z-z')} = e^{-i\phi(z,z')} \tag{11}$$

Hence $2\phi(z,z') = 2\pi n$, $n \in \mathbb{Z}$. Since for $z = z'$ we have $\phi(z,z') = 0$ and ϕ is continuous in z, z', we get $\phi(z,z') = 0$ for all z, z' i.e.

$$\mathrm{Im}\,\langle z|z'\rangle = \mathrm{Im}\langle S(z)|S(z')\rangle, \tag{12}$$

$$c(z)c(z') = c(z+z'). \tag{13}$$

Problem 2. Show that, for continuous c and S, automorphisms from Problem 4 are unitarily implementable in all irreducible representations satisfying the Criterion.
Hint: Use the von Neumann uniqueness theorem.

Solution: Note that $\pi \circ \alpha$ is again an irreducible representation. Since S and c are continuous, we obtain that

$$z \mapsto \langle \Psi_1|\pi \circ \alpha(W(z))\Psi_2\rangle = c(z)\langle \Psi_1|\pi(W(Sz))\Psi_2\rangle \tag{14}$$

i.e. $\pi \circ \alpha$ satisfies the Criterion. Thus it is unitarily equivalent to the Schroedinger representation π_1 which is in turn unitarily equivalent to π

$$\pi \circ \alpha(W) = U_\alpha \pi_1(W) U_\alpha^{-1} = U_\alpha U \pi(W) U^{-1} U_\alpha^{-1}. \tag{15}$$

Problem 3. Let $\eta : \mathbb{R} \times \mathbb{R} \to S^1$ (where S^1 is the unit circle on the complex plane) be a continuous function, differentiable in the second variable, that satisfies the "cocycle relation":

$$\eta(r, s+t)\eta(s,t) = \eta(r+s,t)\eta(r,s) \tag{16}$$

and

$$\eta(s,0) = \eta(0,t) = 1. \tag{17}$$

Show that "the cocycle is a coboundary" that is

$$\eta(s,t) = \frac{\xi(s)\xi(t)}{\xi(s+t)} \tag{18}$$

for some continuous $\xi : \mathbb{R} \to S^1$.

Hints:

(i) First show that $\tilde{\eta}(s,t) := \frac{\eta(s,t)}{\eta(t,s)}$ satisfies

$$\tilde{\eta}(s, t_1 + t_2) = \tilde{\eta}(s, t_1)\tilde{\eta}(s, t_2), \quad \tilde{\eta}(s_1 + s_2, t) = \tilde{\eta}(s_1, t)\tilde{\eta}(s_2, t). \tag{19}$$

Conclude from this and continuity that $\tilde{\eta}(s,t) = 1$ i.e. η is symmetric ($\eta(s,t) = \eta(t,s)$). Hence, η is differentiable in both variables.

(ii) Write $\eta(s,t) = e^{i\phi(s,t)}$. Show that the symmetry and cocycle relation imply

$$\phi_1(0,t) = \phi_2(t,0), \tag{20}$$
$$\phi_1(s,t) = \phi_1(0, s+t) - \phi_1(0,s), \tag{21}$$
$$\phi_2(s,t) = \phi_2(s+t, 0) - \phi_2(t,0). \tag{22}$$

where $\phi_1(s,t) := \partial_r \phi(r+s,t)|_{r=0}$ and $\phi_2(s,t) = \partial_r \phi(s, t+r)|_{r=0}$.

(iii) Define $f(\varepsilon) := \phi(\varepsilon s, \varepsilon t)$ so that

$$\phi(s,t) = \int_0^1 d\varepsilon \, \partial_\varepsilon f(\varepsilon). \tag{23}$$

Use this representation and (20), (21), (22) to construct $\tilde{\phi}$ s.t. $\xi(s) = e^{-i\tilde{\phi}(s)}$ in (18).

Remark 0.1 *The following solution gives the answer only locally i.e. we will get (18) for $s, t, s+t$ in some neighbourhood of zero. The problem is that a priori we can write $\eta(s,t) = e^{i\phi(s,t)}$, with continuous and differentiable $\phi(s,t)$ only locally. Improvements may follow.*

Solution: First we show that $\eta(s,t) = \eta(t,s)$. Let us define

$$\tilde{\eta}(s,t) = \frac{\eta(s,t)}{\eta(t,s)}. \tag{24}$$

We show that

$$\tilde{\eta}(r, s+t) = \tilde{\eta}(r,s)\tilde{\eta}(r,t), \tag{25}$$
$$\tilde{\eta}(r+s, t) = \tilde{\eta}(r,t)\tilde{\eta}(s,t). \tag{26}$$

In fact, by a repetitive application of (16), we have

$$\begin{aligned}
\tilde{\eta}(r, s+t) &= \frac{\eta(r, s+t)}{\eta(s+t, r)} = \frac{\eta(r, s+t)\eta(s,t)}{\eta(s+t, r)\eta(s,t)} = \frac{\eta(r+s, t)\eta(r,s)}{\eta(s, t+r)\eta(t,r)} \\
&= \tilde{\eta}(r,s)\tilde{\eta}(r,t)\frac{\eta(r+s,t)\eta(s,r)}{\eta(s,t+r)\eta(r,t)} = \tilde{\eta}(r,s)\tilde{\eta}(r,t).
\end{aligned} \tag{27}$$

and analogously for (26). By Stone's theorem, we have

$$\tilde{\eta}(s,t) = e^{i\lambda_1(s)t}, \quad \tilde{\eta}(s,t) = e^{i\lambda_2(t)s}, \tag{28}$$

and by continuity of $\tilde{\eta}$ we can conclude that λ_1, λ_2 are continuous in some neighbourhood of zero. Thus we can write

$$\lambda_1(s)t = \lambda_2(t)s + 2\pi n(s,t), \quad n(s,t) \in \mathbb{Z}. \tag{29}$$

Setting $t, s = 0$ we conclude that $n(0,0) = 0$, thus, by continuity of λ_1, λ_2, it must remain zero in some neighbourhood of zero. There we have

$$\frac{\lambda_1(s)}{s} = \frac{\lambda_2(t)}{t} \tag{30}$$

Setting some fixed $t = t_0$ we conclude that $\lambda_1(s) = cs$, hence in a neighbourhood of zero

$$\tilde{\eta}(s,t) = e^{icst}. \tag{31}$$

Since $\tilde{\eta}(s,s) = 1$ for all s, we get $c = 0$. Hence $\tilde{\eta}(s,t) = 0$ for small s, t and by the group property for all s, t. So we have

$$\eta(s,t) = \eta(t,s). \tag{32}$$

Now we write $\eta(s,t) = e^{i\phi(s,t)}$. Relation (16) gives

$$\phi(r, s+t) + \phi(s,t) - \phi(r+s,t) - \phi(r,s) = 2\pi n(r,s,t). \tag{33}$$

Since we have (17), we can demand that

$$\phi(s,0) = \phi(0,t) = 0. \tag{34}$$

As ϕ is continuous, we then get $n(r,s,t) = 0$. By an analogous argument we get from (32)

$$\phi(s,t) = \phi(t,s). \tag{35}$$

Now we know that ϕ is differentiable and differentiate (35) w.r.t. s at zero. We get

$$\phi_1(0,t) = \phi_2(t,0), \tag{36}$$

where we use the notation $\phi_1(s,t) := \partial_r \phi(r+s,t)|_{r=0}$ and analogously for ϕ_2.
Now we differentiate (33) w.r.t. r. This gives

$$\phi_1(s,t) = \phi_1(0, s+t) - \phi_1(0,s). \tag{37}$$

Similarly, by differentiating (33) w.r.t. t at zero we get

$$\phi_2(r,s) + \phi_2(s,0) - \phi_2(r+s,0) = 0. \tag{38}$$

By renaming variables $(r,s) \mapsto (s,t)$:

$$\phi_2(s,t) = \phi_2(s+t,0) - \phi_2(t,0). \tag{39}$$

Now we consider the function $f(\varepsilon) = \phi(\varepsilon s, \varepsilon t)$. Clearly $f(1) = \phi(s,t)$, $f(0) = 0$ so

$$\phi(s,t) = \int_0^1 d\varepsilon\, \partial_\varepsilon f(\varepsilon) = \int_0^1 d\varepsilon\, (s\phi_1(\varepsilon s, \varepsilon t) + t\phi_2(\varepsilon s, \varepsilon t)). \tag{40}$$

Using (37), (39) we get

$$\begin{aligned}\phi(s,t) &= \int_0^1 d\varepsilon\, \big(s(\phi_1(0,\varepsilon(s+t)) - \phi_1(0,\varepsilon s)) + t(\phi_2(\varepsilon(s+t),0) - \phi_2(\varepsilon t,0))\big) \\ &= \int_0^1 d\varepsilon\, \big((s+t)\phi_1(0,\varepsilon(s+t)) - s\phi_1(0,\varepsilon s) - t\phi_1(0,\varepsilon t)), \end{aligned} \tag{41}$$

where in the last step we used (36). Thus setting

$$\tilde{\phi}(s) := s \int_0^1 d\varepsilon\, \phi_1(0,\varepsilon s) \tag{42}$$

we can write

$$\phi(s,t) = \tilde{\phi}(s+t) - \tilde{\phi}(s) - \tilde{\phi}(t). \tag{43}$$

Hence we obtained that

$$\eta(s,t) = \frac{e^{-i\tilde{\phi}(s)} e^{-i\tilde{\phi}(t)}}{e^{-i\tilde{\phi}(s+t)}}. \tag{44}$$