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Algebraic Quantum Field Theory
Homework Sheet 2

Problem 1. A large class of automorphisms of $\mathcal{W}$ may be obtained via the ansatz

$$\alpha(W(z)) = c(z)W(Sz) \tag{1}$$

where $c(z) \in \mathbb{C}\backslash\{0\}$ and $S : \mathbb{C}^n \to \mathbb{C}^n$ a continuous bijection. Show that the Weyl relations impose the following restrictions on c, S:

$$c(z+z') = c(z)c(z'), \qquad c(-z) = \overline{c(z)}, \qquad |c(z)| = 1, \tag{2}$$

$$S(z+z') = S(z) + S(z'), \quad S(-z) = -S(z), \quad \mathrm{Im}\langle Sz, Sz'\rangle = \mathrm{Im}\langle z, z'\rangle. \tag{3}$$

Note that these relations imply that $z \mapsto S(z)$ is a real-linear map and that S is symplectic.

Problem 2. Show that, for continuous c and S, automorphisms from Problem 2 are unitarily implementable in all irreducible representations satisfying the Criterion.

Hint: Use the von Neumann uniqueness theorem.

Problem 3. Let $\eta : \mathbb{R} \times \mathbb{R} \to S^1$ (where S^1 is the unit circle on the complex plane) be a continuous function, differentiable in the second variable, satisfying the "cocycle relation"

$$\eta(r, s+t)\eta(s,t) = \eta(r+s, t)\eta(r,s) \tag{4}$$

$$\eta(s, 0) = \eta(0, t) = 1. \tag{5}$$

Show that "the cocycle is a coboundary" that is

$$\eta(s,t) = \frac{\xi(s)\xi(t)}{\xi(s+t)} \tag{6}$$

for some continuous $\xi : \mathbb{R} \to S^1$.

Suggested solution strategy for Problem 3:

(i) First show that $\tilde{\eta}(s,t) := \frac{\eta(s,t)}{\eta(t,s)}$ satisfies

$$\tilde{\eta}(s, t_1 + t_2) = \tilde{\eta}(s, t_1)\tilde{\eta}(s, t_2), \quad \tilde{\eta}(s_1 + s_2, t) = \tilde{\eta}(s_1, t)\tilde{\eta}(s_2, t). \tag{7}$$

Conclude from this and continuity that $\tilde{\eta}(s,t) = 1$ i.e. η is symmetric ($\eta(s,t) = \eta(t,s)$). Hence, η is differentiable in both variables.

(ii) Write $\eta(s,t) = e^{i\phi(s,t)}$. Show that the symmetry and cocycle relation imply

$$\phi_1(0,t) = \phi_2(t,0), \tag{8}$$
$$\phi_1(s,t) = \phi_1(0, s+t) - \phi_1(0,s), \tag{9}$$
$$\phi_2(s,t) = \phi_2(s+t, 0) - \phi_2(t,0). \tag{10}$$

where $\phi_1(s,t) := \partial_r \phi(s+r, t)|_{r=0}$ and $\phi_2(s,t) = \partial_r \phi(s, t+r)|_{r=0}$.

(iii) Define $f(\varepsilon) := \phi(\varepsilon s, \varepsilon t)$ so that

$$\phi(s,t) = \int_0^1 d\varepsilon\, \partial_\varepsilon f(\varepsilon). \tag{11}$$

Use this representation and (8), (9), (10) to construct $\tilde{\phi}$ s.t. $\xi(s) = e^{-i\tilde{\phi}(s)}$ in (6).

To be discussed in class: 22.5.2017