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Winter semester 2025/26

Stochastic Differential Equations
Homework Sheet 3 - solutions

Problem 1. Show that

$$\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} dz\{|z|^2\}^2 \mathrm{e}^{-\frac{1}{2}|z|^2} = d(d+2), \tag{1}$$

where $|z|^2 := \sum_{i=1}^d (z_i)^2$. Conclude that for $\mathbb{R}$-valued random variable $Z \sim N(0,\sigma^2)$ we have $E(Z^4) = 3\sigma^4$.

Solution. By Problem 5 of HS2

$$I(a) \ := \ \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{a}{2}|z|^2}\, dz = (2\pi)^{d/2}\, a^{-d/2}, \qquad a>0.$$

Differentiating under the integral,

$$I'(a) \ = \ \int_{\mathbb{R}^d} \big(-\tfrac{1}{2}|z|^2\big)\, \mathrm{e}^{-\frac{a}{2}|z|^2}\, dz,$$

$$I''(a) \ = \ \int_{\mathbb{R}^d} \big(-\tfrac{1}{2}|z|^2\big)^2\, \mathrm{e}^{-\frac{a}{2}|z|^2}\, dz.$$

Hence

$$\int_{\mathbb{R}^d} \big(|z|^2\big)^2\ \mathrm{e}^{-\frac{a}{2}|z|^2}\, dz = 4\, I''(a).$$

From the explicit formula for $I(a)$ above

$$I''(a) \ = \ (2\pi)^{d/2}\, \frac{d}{2}\left(\frac{d}{2}+1\right) a^{-\frac{d}{2}-2}.$$

Therefore

$$\int_{\mathbb{R}^d} \big(|z|^2\big)^2\ \mathrm{e}^{-\frac{a}{2}|z|^2}\, dz = (2\pi)^{d/2}\, d(d+2)\, a^{-\frac{d}{2}-2}.$$

Thus for for $a=1$ we obtain the claim.

Now if $d=1$ and $Z \in N(0,\sigma^2)$, then $Z/\sigma \in N(0,1)$ (see lecture notes, construction of Brownian motion via the CLT). Then

$$E\big[\big(Z/\sigma\big)^4\big] = d(d+2) = 3, \tag{2}$$

hence, $E(Z^4) = 3\sigma^4$.

Problem 2. Let $Y \sim N(0,1)$ and set for some $a > 0$

$$Z = Y\chi_{\{|Y|\le a\}} - Y\chi_{\{|Y|>a\}}. \tag{3}$$

Show that also $Z \sim N(0,1)$. Also, show that $Y+Z$ is not normal, hence $X = (Y,Z)$ is not a multi-normal random variable. (In this exercise non-degenerate (multi-)normal distributions are meant).

Notation: $\chi_{\{|Y|\le a\}}(\omega) := \chi_{\{y\in\mathbb{R}\,:\,|y|\le a\}}(Y(\omega))$. The function $\chi_{\{|Y|>a\}}$ is defined analogously.

Solution. Step 1: Z is standard normal.

Let $\phi_Z(u) = E[\mathrm{e}^{iuZ}]$. Then, we verify that, for any characteristic function χ_A,

$$\mathrm{e}^{iuY\chi_A} = \chi_{A^c} + \mathrm{e}^{iuY}\chi_A, \tag{4}$$

by checking separately for $\omega \in A$ and $\omega \in A^c$. Setting $A = \{|Y| \le a\}$ and $A^c = \{|Y| > a\}$, we have

$$\mathrm{e}^{iuZ} = \mathrm{e}^{iuY\chi_A}\mathrm{e}^{-iuY\chi_{A^c}} = (\chi_{A^c} + \mathrm{e}^{iuY}\chi_A)(\chi_A + \mathrm{e}^{-iuY}\chi_{A^c}) = \mathrm{e}^{iuY}\chi_A + \mathrm{e}^{-iuY}\chi_{A^c}. \tag{5}$$

Hence,

$$\phi_Z(u) = E\left[\mathrm{e}^{iuY}\chi_{\{|Y|\le a\}}\right] + E\left[\mathrm{e}^{-iuY}\chi_{\{|Y|>a\}}\right].$$

Compute the second term explicitly via the density of Y:

$$E\left[\mathrm{e}^{-iuY}\chi_{\{|Y|>a\}}\right] = \int_{|y|>a} \mathrm{e}^{-iuy}\,\frac{\mathrm{e}^{-y^2/2}}{\sqrt{2\pi}}\,dy.$$

With the change of variables $x = -y$ we get

$$\int_{|y|>a} \mathrm{e}^{-iuy}\,\frac{\mathrm{e}^{-y^2/2}}{\sqrt{2\pi}}\,dy = \int_{|x|>a} \mathrm{e}^{iux}\,\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\,dx = E\left[\mathrm{e}^{iuY}\chi_{\{|Y|>a\}}\right].$$

Hence

$$\phi_Z(u) = E\left[\mathrm{e}^{iuY}\chi_{\{|Y|\le a\}}\right] + E\left[\mathrm{e}^{iuY}\chi_{\{|Y|>a\}}\right] = E\left[\mathrm{e}^{iuY}\right] = \mathrm{e}^{-u^2/2}.$$

Therefore $Z \sim N(0,1)$.

Step 2: $Y+Z$ is not normal. By definition,

$$Y + Z = \begin{cases} 2Y, & |Y| \le a, \\ 0, & |Y| > a. \end{cases}$$

If $|Y| \le a$ then $Y \le a$ so $2Y \le 2a$, hence $(Y+Z) > 2a$ cannot occur in this case. If $|Y| > a$ then $Y + Z = 0 \le 2a$, so $(Y+Z) > 2a$ also cannot occur. Therefore

$$P(Y + Z > 2a) = 0.$$

But any non-degenerate Gaussian random variable has an everywhere strictly positive density. Consequently, $Y+Z$ cannot be such a Gaussian.

Step 3: (Y,Z) is not jointly normal. If (Y,Z) were jointly normal (non-degenerate), then every nontrivial linear combination, in particular $Y+Z$, would be normal (non-degenerate), contradicting Step 2. Thus $X = (Y,Z)$ is not multi-normal.

Problem 3. Suppose $X_k : \Omega \to \mathbb{R}^d$ are normal for all k and $X_k \to X$ in $L^2(\Omega)$, i.e.

$$E[|X_k - X|^2] \to 0 \quad \text{as} \quad k \to \infty. \tag{6}$$

Show that X is normal.

Solution. The most direct solution appear to be to write

$$E[\exp(i\langle u, X_k\rangle)] = \exp\left(-\frac{1}{2}\langle u, C_k u\rangle + i\langle u, m_k\rangle\right) \tag{7}$$

and then use the formulas from the lecture expressing C_k and m_k in terms of X_k (as done in class).

Here we give a different argument: First, note that $|\mathrm{e}^{i\langle u,x\rangle} - \mathrm{e}^{i\langle u,x\rangle}| \leq |u| \cdot |x-y|$. This follows, e.g., from

$$\begin{aligned}
|\mathrm{e}^{i\langle u,x\rangle} - \mathrm{e}^{i\langle u,x\rangle}| &= |1 - \mathrm{e}^{i\langle u,(x-y)\rangle}| \\
&= |\int_0^1 \frac{d}{dw}\mathrm{e}^{iw\langle u,(x-y)\rangle} dw| \\
&= |i\langle u, (x-y)\rangle \int_0^1 \mathrm{e}^{iw\langle u,(x-y)\rangle} dw| \\
&\leq |u||x-y|.
\end{aligned} \tag{8}$$

Thus, we have

$$E\left[\left\{ \exp(i\langle u, X_k\rangle) - \exp(i\langle u, X\rangle)\right\}^2\right] \leq |u|^2 E[|X_k - X|^2] \tag{9}$$

as $k \to \infty$. Therefore

$$\begin{aligned}
\big|E[\exp(i\langle u, X_k\rangle)] - E[\exp(i\langle u, X\rangle)]\big| &\leq \int |\exp(i\langle u, X_k\rangle) - \exp(i\langle u, X\rangle)| dP &(10)\\
&\leq (\int |\exp(i\langle u, X_k\rangle) - \exp(i\langle u, X\rangle)|^2 dP)^{1/2} \to 0, &(11)
\end{aligned}$$

where we used the Cauchy-Schwarz inequality. Up to now we have not used that X_k are Gaussian. Actually, we have verified, quite generally, that the L^2-convergence implies the convergence of the characteristic function, pointwise in u.

We conclude that

$$E[\exp(i\langle u, X_k\rangle)] = \exp\left(-\frac{1}{2}\langle u, C_k u\rangle + i\langle u, m_k\rangle\right) \tag{12}$$

converge for any fixed u.

Using this, let us now justify that C_k and m_k converge to some C and m. In fact, since (12) converges, also its absolute value does, hence $\exp(-\frac{1}{2}\langle u, C_k u\rangle)$ must converge, therefore also $\langle u, C_k u\rangle$ must converge. As this holds for all u, by polarization,

$$\langle v_1, C_k v_2\rangle = \frac{1}{4}(\langle (v_1 + v_2), C_k(v_1+v_2)\rangle - \langle (v_1 - v_2), C_k(v_1 - v_2)\rangle) \tag{13}$$

all matrix elements converge. Now it suffices to show that convergence of e^{ium_k} for all u implies convergence of m_k. We proceed by contradiction. First, suppose $m_k \to \infty$ (perhaps along a subsequence). Set $F(u) := \lim_{k\to\infty} \mathrm{e}^{ium_k}$. Then, by the Riemann-Lebesgue lemma and dominated convergence, for any $f \in L^1(\mathbb{R}^d)$

$$0 = \lim_{k\to\infty} \int \mathrm{e}^{iu\cdot m_k} f(u)du = \int F(u)f(u)du. \tag{14}$$

But F is non-zero since $F(0) = 1$ and we can choose e.g. $f(u) = \overline{F}(u)\mathrm{e}^{-|u|^2}$ to ensure that the integral (14) is non-zero and finite, which is a contradiction. Now suppose that m_k has two (or more) finite accumulation points. Choose any $g \in L^1(\mathbb{R}^d)$ s.t, $\hat{g}$ takes different values near these accumulation points. Then

$$\int \mathrm{e}^{ium_k} g(u)du = \hat{g}(m_k). \tag{15}$$

By taking the limit of both sides the l.h.s. converges by dominated convergence and the r.h.s. does not as $k \mapsto \hat{g}(m_k)$ jumps between the two accumulation points. Again, a contradiction.

Since $C_k = C_k^T$ for all k, also the limiting matrix is symmetric. Similarly, since for any $u \in \mathbb{R}^d$ we have $\langle u, C_k u\rangle \geq 0$ and the inequality $\geq$ survives limits, we conclude that C has positive eigenvalues.

But we cannot conclude that a sequence of non-degenerate Gaussians converges to a non-degenerate Gaussian, since the inequality $>$ may become $\geq$ in the limit. Actually, such a scenario can easily happen under the assumptions of this problem: Let Z be any non-degenerate Gaussian, set $X_k = \frac{1}{k}Z$ so that it trivially converges in $L^2(\Omega)$ to $X \equiv 0$. Note that $X \equiv 0$ has the Dirac delta distribution hence it is degenerate Gaussian.

Problem 4. Let $\{X_n\}_{n\geq 1}$ and X be random variables on a common probability space. We say that X_n *converges in probability* to X, written $X_n \xrightarrow{P} X$, if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0.$$

Show that if $X_n \to X$ in $L^2(\Omega)$, i.e.

$$E[(X_n - X)^2] \xrightarrow[n\to\infty]{} 0,$$

then $X_n \xrightarrow{P} X$. Hint: Use the Chebyshev's inequality from HS2, Problem 3.

Solution. Will be discussed in class on 07.11.2025 and posted afterwards.

Problem 5. Let $\{B_t\}_{t\in\mathbb{R}_+}$ be d-dimensional Brownian motion, $d \geq 3$, starting at x. Let $K \subset \mathbb{R}^d$ be a bounded Borel set and $T > 0$. Then the random variable

$$\Omega \ni \omega \mapsto \int_0^T \chi_{\{\tilde{\omega}\in\Omega : B_t(\tilde{\omega})\in K\}}(\omega)dt \tag{16}$$

denoted in short-hand notation

$$\int_0^T \chi_{\{B_t \in K\}} dt = \lim_{|\Pi| \to 0} \sum_{i=1}^{n} \chi_{\{B_{\xi_i} \in K\}} \Delta t_i \tag{17}$$

is the time this stochastic process spends in K up to the 'time horizon' T. (This is more clear from the Riemann sum representation, where the sum counts only the time-intervals Δt_i for which the Brownian motion B_{ξ_i}, $\xi_i \in [t_{i-1} t_i]$, is in K). Show that

$$\lim_{T \to \infty} E^x \left[\int_0^T \chi_{\{B_t \in K\}} dt \right] = \int_K G(x, y) dy, \tag{18}$$

where $G(x, y) := \frac{c_d}{|x-y|^{d-2}}$ for some $c_d > 0$.

Solution. Will be discussed in class on 07.11.2025 and posted afterwards.

To be discussed in class: 24.10.2025