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Winter semester 2025/26

Stochastic Differential Equations
Homework Sheet 3 - solutions

Problem 1. Show that

$$\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} dz \big\{|z|^2\big\}^2 \mathrm{e}^{-\frac{1}{2}|z|^2} = d(d+2), \tag{1}$$

where $|z|^2 := \sum_{i=1}^d (z_i)^2$. Conclude that for $\mathbb{R}$-valued random variable $Z \sim N(0,\sigma^2)$ we have $E(Z^4) = 3\sigma^4$.

Solution. By Problem 5 of HS2

$$I(a) \ := \ \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{a}{2}|z|^2}\, dz = (2\pi)^{d/2}\, a^{-d/2}, \qquad a > 0.$$

Differentiating under the integral,

$$I'(a) \ = \ \int_{\mathbb{R}^d} \big(-\tfrac{1}{2}|z|^2\big)\, \mathrm{e}^{-\frac{a}{2}|z|^2}\, dz,$$

$$I''(a) \ = \ \int_{\mathbb{R}^d} \big(-\tfrac{1}{2}|z|^2\big)^2\, \mathrm{e}^{-\frac{a}{2}|z|^2}\, dz.$$

Hence

$$\int_{\mathbb{R}^d} \big(|z|^2\big)^2\ \mathrm{e}^{-\frac{a}{2}|z|^2}\, dz = 4\, I''(a).$$

From the explicit formula for $I(a)$ above

$$I''(a) \ = \ (2\pi)^{d/2}\, \frac{d}{2}\left(\frac{d}{2}+1\right) a^{-\frac{d}{2}-2}.$$

Therefore

$$\int_{\mathbb{R}^d} \big(|z|^2\big)^2\ \mathrm{e}^{-\frac{a}{2}|z|^2}\, dz = (2\pi)^{d/2}\, d(d+2)\, a^{-\frac{d}{2}-2}.$$

Thus for for $a = 1$ we obtain the claim.

Now if $d = 1$ and $Z \in N(0,\sigma^2)$, then $Z/\sigma \in N(0,1)$ (see lecture notes, construction of Brownian motion via the CLT). Then

$$E\big[\big(Z/\sigma\big)^4\big] = d(d+2) = 3, \tag{2}$$

hence, $E(Z^4) = 3\sigma^4$.

Problem 2. Let $Y \sim N(0,1)$ and set for some $a > 0$

$$Z = Y\chi_{\{|Y|\leq a\}} - Y\chi_{\{|Y|>a\}}. \tag{3}$$

Show that also $Z \sim N(0,1)$. Also, show that $Y+Z$ is not normal, hence $X = (Y,Z)$ is not a multi-normal random variable. (In this exercise non-degenerate (multi-)normal distributions are meant).

Notation: $\chi_{\{|Y|\leq a\}}(\omega) := \chi_{\{y\in\mathbb{R}\,:\,|y|\leq a\}}(Y(\omega))$. The function $\chi_{\{|Y|>a\}}$ is defined analogously.

Solution. Step 1: Z is standard normal.

Let $\phi_Z(u) = E[\mathrm{e}^{iuZ}]$. Then, we verify that, for any characteristic function χ_A,

$$\mathrm{e}^{iuY\chi_A} = \chi_{A^c} + \mathrm{e}^{iuY}\chi_A, \tag{4}$$

by checking separately for $\omega \in A$ and $\omega \in A^c$. Setting $A = \{|Y| \leq a\}$ and $A^c = \{|Y| > a\}$, we have

$$\mathrm{e}^{iuZ} = \mathrm{e}^{iuY\chi_A}\mathrm{e}^{-iuY\chi_{A^c}} = (\chi_{A^c} + \mathrm{e}^{iuY}\chi_A)(\chi_A + \mathrm{e}^{-iuY}\chi_{A^c}) = \mathrm{e}^{iuY}\chi_A + \mathrm{e}^{-iuY}\chi_{A^c}. \tag{5}$$

Hence,

$$\phi_Z(u) = E\left[\mathrm{e}^{iuY}\chi_{\{|Y|\leq a\}}\right] + E\left[\mathrm{e}^{-iuY}\chi_{\{|Y|>a\}}\right].$$

Compute the second term explicitly via the density of Y:

$$E\left[\mathrm{e}^{-iuY}\chi_{\{|Y|>a\}}\right] = \int_{|y|>a} \mathrm{e}^{-iuy}\,\frac{\mathrm{e}^{-y^2/2}}{\sqrt{2\pi}}\,dy.$$

With the change of variables $x = -y$ we get

$$\int_{|y|>a} \mathrm{e}^{-iuy}\,\frac{\mathrm{e}^{-y^2/2}}{\sqrt{2\pi}}\,dy = \int_{|x|>a} \mathrm{e}^{iux}\,\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\,dx = E\left[\mathrm{e}^{iuY}\chi_{\{|Y|>a\}}\right].$$

Hence

$$\phi_Z(u) = E\left[\mathrm{e}^{iuY}\chi_{\{|Y|\leq a\}}\right] + E\left[\mathrm{e}^{iuY}\chi_{\{|Y|>a\}}\right] = E\left[\mathrm{e}^{iuY}\right] = \mathrm{e}^{-u^2/2}.$$

Therefore $Z \sim N(0,1)$.

Step 2: $Y+Z$ is not normal. By definition,

$$Y + Z = \begin{cases} 2Y, & |Y| \leq a, \\ 0, & |Y| > a. \end{cases}$$

If $|Y| \leq a$ then $Y \leq a$ so $2Y \leq 2a$, hence $(Y+Z) > 2a$ cannot occur in this case. If $|Y| > a$ then $Y + Z = 0 \leq 2a$, so $(Y+Z) > 2a$ also cannot occur. Therefore

$$P(Y + Z > 2a) = 0.$$

But any non-degenerate Gaussian random variable has an everywhere strictly positive density. Consequently, $Y+Z$ cannot be such a Gaussian.

Step 3: (Y,Z) is not jointly normal. If (Y,Z) were jointly normal (non-degenerate), then every nontrivial linear combination, in particular $Y+Z$, would be normal (non-degenerate), contradicting Step 2. Thus $X = (Y,Z)$ is not multi-normal.

Problem 3. Suppose $X_k : \Omega \to \mathbb{R}^d$ are normal for all k and $X_k \to X$ in $L^2(\Omega)$, i.e.

$$E[|X_k - X|^2] \to 0 \quad \text{as} \quad k \to \infty. \tag{6}$$

Show that X is normal.

Solution. The most direct solution appear to be to write

$$E[\exp(i\langle u, X_k\rangle)] = \exp\left(-\frac{1}{2}\langle u, C_k u\rangle + i\langle u, m_k\rangle\right) \tag{7}$$

and then use the formulas from the lecture expressing C_k and m_k in terms of X_k (as done in class).

Here we give a different argument: First, note that $|\mathrm{e}^{i\langle u,x\rangle} - \mathrm{e}^{i\langle u,x\rangle}| \leq |u| \cdot |x - y|$. This follows, e.g., from

$$\begin{aligned}
|\mathrm{e}^{i\langle u,x\rangle} - \mathrm{e}^{i\langle u,x\rangle}| &= |1 - \mathrm{e}^{i\langle u,(x-y)\rangle}| \\
&= |\int_0^1 \frac{d}{dw}\mathrm{e}^{iw\langle u,(x-y)\rangle} dw| \\
&= |i\langle u, (x-y)\rangle \int_0^1 \mathrm{e}^{iw\langle u,(x-y)\rangle} dw| \\
&\leq |u||x-y|.
\end{aligned} \tag{8}$$

Thus, we have

$$E\left[\left\{ \exp(i\langle u, X_k\rangle) - \exp(i\langle u, X\rangle)\right\}^2\right] \leq |u|^2 E[|X_k - X|^2] \tag{9}$$

as $k \to \infty$. Therefore

$$\big|E[\exp(i\langle u, X_k\rangle)] - E[\exp(i\langle u, X\rangle)]\big| \leq \int |\exp(i\langle u, X_k\rangle) - \exp(i\langle u, X\rangle)| dP \tag{10}$$

$$\leq (\int |\exp(i\langle u, X_k\rangle) - \exp(i\langle u, X\rangle)|^2 dP)^{1/2} \to 0, \tag{11}$$

where we used the Cauchy-Schwarz inequality. Up to now we have not used that X_k are Gaussian. Actually, we have verified, quite generally, that the L^2-convergence implies the convergence of the characteristic function, pointwise in u.

We conclude that

$$E[\exp(i\langle u, X_k\rangle)] = \exp\left(-\frac{1}{2}\langle u, C_k u\rangle + i\langle u, m_k\rangle\right) \tag{12}$$

converge for any fixed u.

Using this, let us now justify that C_k and m_k converge to some C and m. In fact, since (12) converges, also its absolute value does, hence $\exp(-\frac{1}{2}\langle u, C_k u\rangle)$ must converge, therefore also $\langle u, C_k u\rangle$ must converge. As this holds for all u, by polarization,

$$\langle v_1, C_k v_2\rangle = \frac{1}{4}(\langle (v_1 + v_2), C_k(v_1 + v_2)\rangle - \langle (v_1 - v_2), C_k(v_1 - v_2)\rangle) \tag{13}$$

all matrix elements converge. Now it suffices to show that convergence of e^{ium_k} for all u implies convergence of m_k. We proceed by contradiction. First, suppose $m_k \to \infty$ (perhaps along a subsequence). Set $F(u) := \lim_{k\to\infty} \mathrm{e}^{ium_k}$. Then, by the Riemann-Lebesgue lemma and dominated convergence, for any $f \in L^1(\mathbb{R}^d)$

$$0 = \lim_{k\to\infty} \int \mathrm{e}^{iu\cdot m_k} f(u)du = \int F(u)f(u)du. \tag{14}$$

But F is non-zero since $F(0) = 1$ and we can choose e.g. $f(u) = \overline{F}(u)\mathrm{e}^{-|u|^2}$ to ensure that the integral (14) is non-zero and finite, which is a contradiction. Now suppose that m_k has two (or more) finite accumulation points. Choose any $g \in L^1(\mathbb{R}^d)$ s.t, $\hat{g}$ takes different values near these accumulation points. Then

$$\int \mathrm{e}^{ium_k} g(u)du = \hat{g}(m_k). \tag{15}$$

By taking the limit of both sides the l.h.s. converges by dominated convergence and the r.h.s. does not as $k \mapsto \hat{g}(m_k)$ jumps between the two accumulation points. Again, a contradiction.

Since $C_k = C_k^T$ for all k, also the limiting matrix is symmetric. Similarly, since for any $u \in \mathbb{R}^d$ we have $\langle u, C_k u\rangle \geq 0$ and the inequality $\geq$ survives limits, we conclude that C has positive eigenvalues.

But we cannot conclude that a sequence of non-degenerate Gaussians converges to a non-degenerate Gaussian, since the inequality $>$ may become $\geq$ in the limit. Actually, such a scenario can easily happen under the assumptions of this problem: Let Z be any non-degenerate Gaussian, set $X_k = \frac{1}{k}Z$ so that it trivially converges in $L^2(\Omega)$ to $X \equiv 0$. Note that $X \equiv 0$ has the Dirac delta distribution hence it is degenerate Gaussian.

Problem 4. Let $\{X_n\}_{n\geq 1}$ and X be random variables on a common probability space. We say that X_n *converges in probability* to X, written $X_n \xrightarrow{P} X$, if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0.$$

Show that if $X_n \to X$ in $L^2(\Omega)$, i.e.

$$E[(X_n - X)^2] \xrightarrow[n\to\infty]{} 0,$$

then $X_n \xrightarrow{P} X$. Hint: Use the Chebyshev's inequality from HS2, Problem 3.

Solution. To prove (b), fix $\varepsilon > 0$. By the Chebyshev's inequality,

$$P(|X_n - X| > \varepsilon) \ \leq \ \frac{E[(X_n - X)^2]}{\varepsilon^2}.$$

If $E[(X_n - X)^2] \to 0$ as $n \to \infty$, then the right-hand side tends to 0 for each fixed $\varepsilon > 0$, hence

$$\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0,$$

which is exactly $X_n \xrightarrow{P} X$.

Side remark: The topology of convergence in probability comes from the metric

$$d_P(X,Y) = P(\min(|X-Y|,1)) \tag{16}$$

(Ky Fan metric) The space of measurable functions equipped with this metric is denoted (L^0, d_P). It is a complete topological vector space. However, there are no non-zero linear functionals on this space, which would be continuous in this topology (trivial topological dual), if P is non-atomic (Theorem 13.41 of C.D. Aliprantis, K.C. Border *Infinite Dimensional Analysis. A Hitchhiker's Guide).* Hence, there is also no norm generating the topology.

Problem 5. Let $\{B_t\}_{t\in\mathbb{R}_+}$ be d-dimensional Brownian motion, $d \geq 3$, starting at x. Let $K \subset \mathbb{R}^d$ be a bounded Borel set and $T > 0$. Then the random variable

$$\Omega \ni \omega \mapsto \int_0^T \chi_{\{\tilde{\omega}\in\Omega: B_t(\tilde{\omega})\in K\}}(\omega)dt = \int_0^T \chi_K(B_t(\omega))dt \tag{17}$$

denoted in short-hand notation

$$\int_0^T \chi_{\{B_t\in K\}}dt = \lim_{|\Pi|\to 0}\sum_{i=1}^n \chi_{\{B_{\xi_i}\in K\}}\Delta t_i \tag{18}$$

is the time this stochastic process spends in K up to the 'time horizon' T. (This is more clear from the Riemann sum representation, where the sum counts only the time-intervals Δt_i for which the Brownian motion B_{ξ_i}, $\xi_i \in [t_{i-1}t_i]$, is in K). Show that

$$\lim_{T\to\infty} E^x\left[\int_0^T \chi_{\{B_t\in K\}}dt\right] = \int_K G(x,y)dy, \tag{19}$$

where $G(x,y) := \frac{c_d}{|x-y|^{d-2}}$ for some $c_d > 0$.

Solution. Let $\{B_t\}_{t\geq 0}$ be the d-dimensional Brownian motion, $d \geq 3$, starting at $x \in \mathbb{R}^d$. Recall

$$p(t,x,y) := (2\pi t)^{-d/2}\,\mathrm{e}^{-\frac{|x-y|^2}{2t}}, \qquad t>0,\ x,y\in\mathbb{R}^d.$$

The expected time the process spends in K up to time T is

$$E^x\left[\int_0^T \chi_{\{B_t\in K\}}\,dt\right] = \int_0^T P^x(B_t\in K)\,dt = \int_0^T\int_K p(t,x,y)\,dy\,dt,$$

where the Fubini theorem justifies the exchange of integrals.

Letting $T\to\infty$ and applying the Fubini and monotone convergence theorem (for an increasing sequence of positive integrands we can exchange the limit with integral) gives

$$\lim_{T\to\infty} E^x\left[\int_0^T \chi_{\{B_t\in K\}}\,dt\right] = \int_K\left(\int_0^\infty p(t,x,y)\,dt\right)dy = \int_K G(x,y)\,dy,$$

where we defined the Green function:

$$G(x,y) = \int_0^\infty p(t,x,y)\,dt.$$

- **Computation of $G(x,y)$.** Let $r = |x-y|$. Then

$$G(x,y) = (2\pi)^{-d/2} \int_0^\infty t^{-d/2}\, \mathrm{e}^{-\frac{r^2}{2t}}\, dt.$$

Using the substitution $u = \frac{r^2}{2t}$, i.e. $t = \frac{r^2}{2u}$ and $dt = -\frac{r^2}{2u^2} du$, we obtain

$$G(x,y) = (2\pi)^{-d/2}\, r^{2-d} \int_0^\infty 2^{\frac{d}{2}-1} u^{\frac{d}{2}-2}\, \mathrm{e}^{-u}\, du = \frac{\Gamma(\frac{d}{2}-1)}{2\pi^{d/2}}\, r^{2-d},$$

where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$. Equivalently,

$$G(x,y) = \frac{2}{(d-2)\,|S^{d-1}|}\, |x-y|^{2-d},$$

since $|S^{d-1}| = \frac{2\pi^{d/2}}{\Gamma(d/2)}$.

- **Conclusion.** For $d \geq 3$,

$$\lim_{T\to\infty} E^x \left[\int_0^T \chi_{\{B_t \in K\}}\, dt \right] = \int_K G(x,y)\, dy = \int_K \frac{c_d}{|x-y|^{d-2}}\, dy,$$

with $c_d = \frac{\Gamma(\frac{d}{2}-1)}{2\pi^{d/2}} > 0$.

- **Side remark:** G is an example of a Green function (also called a fundamental solution in the PDE theory). Schematically, they satisfy:

$$[\text{Differential operator}][\text{Green function}] = [\text{Dirac delta}]. \tag{20}$$

In our case, it holds

$$-\tfrac{1}{2}\Delta_y G(x,y) = \delta_x(y) \quad \text{in} \quad S'(\mathbb{R}^d). \tag{21}$$

In fact, since $G(x,y) = \int_0^\infty p(t,x,y)\, dt$ and $\partial_t p(t,x,y) = \frac{1}{2}\Delta_y p(t,x,y)$, we have

$$-\frac{1}{2}\Delta_y G(x,y) = \int_0^\infty (-\partial_t) p(t,x,y)\, dt = p(0,x,y) - p(\infty,x,y) = \delta_x(y). \tag{22}$$

Actually, one can also show (21) by using Fourier transforms: Fix $d \geq 3$ and define

$$G(x,y) = \frac{c_d}{|x-y|^{d-2}}, \qquad x,y \in \mathbb{R}^d.$$

Fourier transform convention.

We use

$$\widehat{f}(u) = \int_{\mathbb{R}^d} e^{iy\cdot u}\, f(y)\, dy, \qquad f(y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-iy\cdot u}\, \widehat{f}(u)\, du.$$

Then

$$\widehat{\partial_{y_j} f}(u) = iu_j \widehat{f}(u), \qquad \widehat{\Delta_y f}(u) = -|u|^2 \widehat{f}(u), \qquad \widehat{\delta_x}(u) = e^{ix\cdot u}.$$

- **Step 1: Fourier transform of $G(x,\cdot)$.** By translation invariance of the Fourier transform,

$$\widehat{G(x,\cdot)}(u) = c_d\, e^{ix\cdot u}\, \widehat{|\cdot|^{-(d-2)}}(u).$$

  A standard Riesz–potential identity gives

$$\widehat{|y|^{-(d-2)}}(u) = C_d\, |u|^{-2}, \qquad C_d = \frac{4\pi^{d/2}}{\Gamma\left(\frac{d-2}{2}\right)}.$$

  Therefore,

$$\widehat{G(x,\cdot)}(u) = c_d\, C_d\, e^{ix\cdot u}\, |u|^{-2}.$$

- **Step 2: Apply $-\frac{1}{2}\Delta_y$.** Using $\widehat{\Delta_y f} = -|u|^2 \widehat{f}$,

$$-\tfrac{1}{2}\widehat{\Delta_y G(x,\cdot)}(u) = \tfrac{1}{2}|u|^2 \widehat{G(x,\cdot)}(u) = \tfrac{1}{2} c_d C_d\, e^{ix\cdot u}.$$

  We want this to equal $\widehat{\delta_x}(u) = e^{ix\cdot u}$, hence we require

$$\tfrac{1}{2} c_d C_d = 1 \quad\Longleftrightarrow\quad c_d = \frac{2}{C_d}.$$

- **Step 3: Simplify the constant.** Since the surface area of the unit sphere in $\mathbb{R}^d$ is

$$|S^{d-1}| = \frac{2\pi^{d/2}}{\Gamma(d/2)} \quad \text{and} \quad \Gamma\left(\tfrac{d}{2}\right) = \tfrac{d-2}{2}\,\Gamma\left(\tfrac{d-2}{2}\right),$$

  we obtain

$$\frac{2}{C_d} = \frac{2}{\frac{4\pi^{d/2}}{\Gamma((d-2)/2)}} = \frac{\Gamma((d-2)/2)}{2\pi^{d/2}} = \frac{2}{(d-2)\,|S^{d-1}|}.$$

  Thus,

$$c_d = \frac{2}{(d-2)\,|S^{d-1}|}.$$

- **Conclusion.** With this choice of c_d, we have

$$-\tfrac{1}{2}\widehat{\Delta_y G(x,\cdot)}(u) = e^{ix\cdot u} = \widehat{\delta_x}(u),$$

  and by injectivity of the Fourier transform on $S'(\mathbb{R}^d)$,

$$-\tfrac{1}{2}\Delta_y G(x,y) = \delta_x(y)$$

  in the sense of distributions.

There is a random walk version of Problem 5, known as Polya theorem, which we discuss here briefly as additional material:

Theorem 0.1. The simple symmetric random walk $\{X_n\}_{n\geq 0}$ on $\mathbb{Z}^d$ (one step to each of the $2d$ nearest neighbors with prob. $1/(2d)$) starting at 0:

(a) Is recurrent if $d = 1, 2$. That is P(the walk returns to the origin infinitely often) = 1.

(b) Is transient if $d \geq 3$. That is P(the walk returns to the origin infinitely often) = 0.

Remark 0.2. In low dimensions the space is so tight that there is nowhere to escape. In higher dimensions there are so many different directions that the walk escapes eventually.

Proof for $d = 1$**:** Let x_n be the position of random walker after n steps on $\mathbb{Z}$, like in the first lecture. Denote by $N = \sum_{n=0}^{\infty} \chi(x_n = 0)$ the random variable which counts the number of visits to zero. Then

$$G(0,0) := \sum_{n=0}^{\infty} P(x_n = 0) = E\big[N\big]. \tag{23}$$

We define

$$u := P(\text{the walk returns to 0 at some time } n \geq 1) \tag{24}$$

and we want to show $u = 1$. Then the walk is recurrent, because after it comes back to zero and restarts, it will again return to zero with probability one and so on.

Let $R := N - 1$ be the number of returns to zero after time 0. The process of "returns to zero" behaves like a sequence of independent trials:

- with probability u failure to escape.
- with probability $1 - u$ escapes.

Thus R follows a geometric distribution (Geom$(1-u)$ - number of Bernoulli trials until the first "success" - in our case first escape)

$$P(R = k) = u^k(1-u), \quad k = 0, 1, 2, \ldots \tag{25}$$

and

$$E[R] = \sum_{k=0}^{\infty} k u^k (1-u) = \frac{u}{1-u}. \tag{26}$$

Hence

$$E[N] = 1 + E[R] = \frac{1}{1-u}. \tag{27}$$

Consequently, the walk is recurrent iff $G(0,0)$ diverges. We note that

$$G(0,0) \geq \sum_{n=0}^{\infty} P(x_{2n} = 0) = \sum_{n=0}^{\infty} P[0, 2n] = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^{2n}} = \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} \frac{1}{2^{2n}}, \tag{28}$$

where we used $P[k,n] := P(R_n = (n+k)/2) = \binom{n}{\frac{1}{2}(n+k)}\frac{1}{2^n}$. The last series is divergent as one can check using the Stirling formula

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \mathrm{e}^{\frac{1}{12n+1}} < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \mathrm{e}^{\frac{1}{12n}}. \tag{29}$$

In fact

$$\frac{(2n)!}{(n!)^2}\frac{1}{2^{2n}} \geq \frac{1}{2^{2n}}\frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}\mathrm{e}^{\frac{1}{24n+1}}}{(\sqrt{2\pi n})^2\left(\frac{n}{e}\right)^{2n}\mathrm{e}^{\frac{2}{12n}}} = \frac{1}{\sqrt{\pi n}}\mathrm{e}^{\frac{1}{24n+1}-\frac{2}{12n}} \geq \frac{1}{2}\frac{1}{\sqrt{\pi n}} \tag{30}$$

for sufficiently large n so (28) diverges. □

To be discussed in class: 24.10.2025