

Stochastic Differential Equations

Homework Sheet 3

Problem 1. Show that

$$\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} dz \{ |z|^2 \}^2 e^{-\frac{1}{2}|z|^2} = d(d+2), \quad (1)$$

where $|z|^2 := \sum_{i=1}^d (z_i)^2$. Conclude that for $\mathbb{R}$ -valued random variable $Z \sim N(0, \sigma^2)$ we have $E(Z^4) = 3\sigma^2$.

Problem 2. Let $Y \sim N(0, 1)$ and set for some $a > 0$

$$Z = Y \chi_{\{|Y| \leq a\}} - Y \chi_{\{|Y| > a\}}. \quad (2)$$

Show that also $Z \sim N(0, 1)$. Also, show that $Y + Z$ is not normal, hence $X = (Y, Z)$ is not a multi-normal random variable. (In this exercise non-degenerate (multi-)normal distributions are meant).

Notation: $\chi_{\{|Y| \leq a\}}$ denotes the following characteristic function:

$$\Omega \ni \omega \mapsto \chi_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A \end{cases} \quad (3)$$

for $A := \{ \tilde{\omega} \in \Omega : |Y(\tilde{\omega})| \leq a \}$. The function $\chi_{\{|Y| > a\}}$ is defined analogously.

Problem 3. Suppose $X_k : \Omega \rightarrow \mathbb{R}^d$ are normal for all k and $X_k \rightarrow X$ in $L^2(\Omega)$, i.e.

$$E[|X_k - X|^2] \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (4)$$

Show that X is normal.

Problem 4. Let $\{X_n\}_{n \geq 1}$ and X be random variables on a common probability space. We say that X_n *converges in probability* to X , written $X_n \xrightarrow{P} X$, if for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} P(|X_n - X| > \varepsilon) = 0.$$

Show that if $X_n \rightarrow X$ in $L^2(\Omega)$, then $X_n \xrightarrow{P} X$. Hint: Use the Chebyshev's inequality from HS2, Problem 3.

Problem 5. Let $\{B_t\}_{t \in \mathbb{R}_+}$ be d -dimensional Brownian motion, $d \geq 3$, starting at x . Let $K \subset \mathbb{R}^d$ be a bounded Borel set and $T > 0$. Then the random variable

$$\Omega \ni \omega \mapsto \int_0^T \chi_{\{\tilde{\omega} \in \Omega: B_t(\tilde{\omega}) \in K\}}(\omega) dt \quad (5)$$

denoted in short-hand notation

$$\int_0^T \chi_{\{B_t \in K\}} dt \quad (6)$$

is the time this stochastic process spends in K up to the ‘time horizon’ T . Show that

$$\lim_{T \rightarrow \infty} E^x \left[\int_0^T \chi_{\{B_t \in K\}} dt \right] = \int_K G(x, y) dy, \quad (7)$$

where $G(x, y) := \frac{c_d}{|x-y|^{d-2}}$ for some $c_d > 0$.

To be discussed in class: 24.10.2025