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Algebraic Quantum Field Theory
Homework Sheet 3

Problem 1. We define $\mathcal{V}$ as the polynomial $*$-algebra generated by $V(z)$, $(z \in \mathbb{C}^n)$, satisfying the relations $V(z)^* = V(-z)$ and

$$V(z)V(z') = e^{i\mathrm{Im}\langle z,z'\rangle} V(z')V(z). \tag{1}$$

Show that given any representation π of $\mathcal{W}$, we can use the prescription $\pi_1(V(z)) := \pi(W(z))$ to obtain a well-defined representation of $\mathcal{V}$.

Solution: The map π_1 is prescribed only for the generating elements $V(z)$ of the free algebra $\mathcal{V}_0$. Its extension by linearity and multiplicativity to all of $\mathcal{V}_0$, satisfies these properties by definition. Similarly $*$-symmetry follows from the corresponding property of π. It remains to show that π_1 restricts to $\mathcal{V}$, i.e. where it is well-defined regarding relation (1). This follows from

$$\begin{aligned}
\pi_1\left(V(z)V(z')\right) &= \pi_1\left(V(z)\right)\pi_1\left(V(z')\right) = \pi\left(W(z)\right)\pi\left(W(z')\right) = \pi\left(W(z)W(z')\right) \\
&= \pi\left(e^{i/2\mathrm{Im}\langle z,z'\rangle} W(z+z')\right) = \pi\left(e^{i\mathrm{Im}\langle z,z'\rangle} W(z')W(z)\right) \\
&= e^{i\mathrm{Im}\langle z,z'\rangle}\pi\left(W(z')\right)\pi\left(W(z)\right) = e^{i\mathrm{Im}\langle z,z'\rangle}\pi_1\left(V(z')\right)\pi_1\left(V(z)\right) \\
&= \pi_1\left(e^{i\mathrm{Im}\langle z,z'\rangle} V(z')V(z)\right).
\end{aligned}$$

From this it follows that the ideal $\mathcal{I}$ generated by $V(z)V(z') - e^{i\mathrm{Im}\langle z,z'\rangle}V(z')V(z)$, $z, z' \in \mathbb{C}^n$ is mapped to zero, so that $\pi_1(V + \mathcal{I}) := \pi_1(V)$ is well-defined for $V + \mathcal{I} \in \mathcal{V} = \mathcal{V}_0/\mathcal{I}$.

Problem 2. Let $R(\lambda, z)$ be as in the definition of the pre-resolvent algebra. Show that in any $*$-representation π

$$\|\pi(R(\lambda, z))\| \le \frac{1}{\lambda}, \tag{2}$$

and therefore the norm of any element of the pre-resolvent algebra is bounded.

Solution: The relevant relations are

$$R(\lambda, z)^* \;=\; R(-\lambda, z) \tag{3}$$
$$R(\lambda, z) - R(\mu, z) \;=\; i(\mu - \lambda)R(\lambda, z)R(\mu, z). \tag{4}$$

Using $R(\lambda, z)^* = R(-\lambda, z)$ and the C^*-property of the norm, we get

$$\begin{aligned}
2|\lambda|\|\pi(R(\lambda, z))\|^2 \;&=\; \|\pi\big(2\lambda R(\lambda, z)R(\lambda, z)^*\big)\| \\
&=\; \|\pi\big(R(\lambda, z) - R(\lambda, z))^*\big)\| \le 2\|\pi(R(\lambda, z))\|.
\end{aligned} \tag{5}$$

If $\|\pi(R(\lambda,z))\| \neq 0$ we can cancel and get $\|\pi(R(\lambda,z))\| \leq \frac{1}{\lambda}$. If $\|\pi(R(\lambda,z))\| = 0$ then obviously this is less than $1/\lambda$.

Problem 3. Let $\mathcal{A} \subset B(\mathcal{H})$ be a norm-closed unital $*$-algebra of operators (i.e. a concrete C^*-algebra) which acts irreducibly on a separable Hilbert space $\mathcal{H}$ (i.e. $\mathcal{A}' = \mathbb{C}I$). Suppose that $\mathcal{A}$ contains one non-zero compact operator C. Show that then it contains all of them. Hints:

(i) Note that you can assume that $C = C^*$.

(ii) Recall that the spectrum of a self-adjoint compact operator is discrete and has at most one accumulation point (at zero). Non-zero eigenvalues have finite multiplicity.

(iii) If A is a self-adjoint element of a C^*-algebra then also $f(A)$ is an element of this C^*-algebra for any continuous function f.

(iv) Exploiting the above information show that $\mathcal{A}$ contains a finite-dimensional projection. Using the bicommutant theorem and irreducibility conclude that it contains a one-dimensional projection.

(v) Using irreducibility again show that it contains all compact operators.

Solution: Since $\mathcal{A}$ is a $*$-algebra, we have that $\frac{1}{2}(C+C^*)$ and $\frac{1}{2i}(C-C^*)$ are in $\mathcal{A}$, and at least one of them must be non-zero. Thus, replacing C with its real or imaginary part, we can assume that it is s.a.

Let λ be a non-zero eigenvalue of C and f a continuous function s.t. $\mathrm{supp} f \cap \mathrm{Sp}\, C = \{\lambda\}$ and $f(\lambda) = 1$. Then $f(C) \in \mathcal{A}$ and it is a finite-dimensional projection $P^\lambda = \sum_{i=1}^N |\Psi_i\rangle\langle\Psi_i|$, Ψ_i mutually orthogonal and $\|\Psi_i\| = 1$.

The bicommutant theorem and irreducibility give that $\mathcal{A}'' = B(\mathcal{H})$ and $\mathcal{A}$ is strongly dense in $B(\mathcal{H})$. In particular there is a sequence A_n s.t.

$$\text{s}-\lim_{n\to\infty} A_n = A := |\Psi_1\rangle\langle\Psi_1| \tag{6}$$

Then, since P^λ is a finite-dimensional projection and $P^\lambda \in \mathcal{A}$

$$\text{n}-\lim_{n\to\infty} A_n P^\lambda = AP^\lambda = A. \tag{7}$$

So $A \in \mathcal{A}$. In fact

$$\|(A_n - A)P_\lambda\| \leq \sum_{i=1}^N \|(A_n - A)\Psi_i\| \to 0. \tag{8}$$

Finally, let $|\Phi\rangle\langle\Phi|$ be any other one dimensional projection. By irreducibility Ψ_1 is cyclic so there is a sequence $B_n \in \mathcal{A}$ s.t. $B_n\Psi_1 \to \Phi$. Hence

$$\text{n}-\lim_{n\to\infty} B_n|\Psi_1\rangle\langle\Psi_1|B_n^* = |\Phi\rangle\langle\Phi| \tag{9}$$

and therefore $|\Phi\rangle\langle\Phi| \in \mathcal{A}$. In fact, this follows from.

$$\begin{aligned}&B_n|\Psi_1\rangle\langle\Psi_1|B_n^* - |\Phi\rangle\langle\Phi| \\ &= (B_n|\Psi_1\rangle - |\Phi\rangle)\langle\Psi_1|B_n^* + |\Phi\rangle(\langle\Psi_1|B_n^* - \langle\Phi|)\end{aligned} \tag{10}$$

Now the proof is concluded by making use of the fact that any compact operator is a norm limit of finite-dimensional projections.

Introduction to Problem 4 : Consider an unbounded operator A on a dense domain $D(A) \subset \mathcal{H}$.

Define the graph of A (denoted $\mathrm{Gr}(A)$) as the set of pairs $(\varphi, A\varphi)$, $\varphi \in D(A)$. This is a subset of $\mathcal{H} \times \mathcal{H}$ which is a Hilbert space with the product:

$$\langle(\varphi_1, \psi_1), (\varphi_2, \psi_2)\rangle = \langle\varphi_1, \varphi_2\rangle + \langle\psi_1, \psi_2\rangle \tag{11}$$

(i) We say that $A, D(A)$ is a closed operator if $\mathrm{Gr}(A)$ is closed.

(ii) We say that A_1 is an extension of A if $\mathrm{Gr}(A_1) \supset \mathrm{Gr}(A)$.

(iii) We say that A is closable if it has a closed extension. The smallest closed extension is called the closure $\overline{A}$.

Define $D(A^*)$ as the set of all $\varphi \in \mathcal{H}$, for which there exists $\eta \in \mathcal{H}$ s.t.

$$\langle A\psi, \varphi\rangle = \langle\psi, \eta\rangle, \quad \psi \in D(A). \tag{12}$$

For such $\varphi \in D(A^*)$ we define $A^*\varphi = \eta$.

(i) We say that A is self-adjoint if $A = A^*$ and $D(A) = D(A^*)$.

(ii) We say that A is symmetric if $D(A) \subset D(A^*)$ and $A\psi = A^*\psi$ for $\psi \in D(A)$.
Fact: Any symmetric operator is closable.

(iii) We say that symmetric A is essentially self adjoint if $\overline{A}$ is self-adjoint.
Fact: If A is essentially self-adjoint then it has exactly one self-adjoint extension.

Problem 4. (a) Show that $N := \mathrm{d}\Gamma(1)$ is an unbounded operator on $\Gamma(\mathfrak{h})$.
(b) Show that for any bounded s.a. operator b on $\mathfrak{h}$ its second quantization $\mathrm{d}\Gamma(b)$ is essentially self adjoint on $\Gamma_{\mathrm{fin}}(\mathfrak{h})$. Hints:

(i) Recall that a symmetric operator T on a domain $D(T)$ is essentially self adjoint if it has a unique self-adjoint extension.

(ii) Essential self-adjointness is equivalent to $(T \pm i)D(T)$ being dense.

Solution: (a) note that $N\Psi^{(n)} = n\Psi^{(n)}$ for any $\Psi^n \in \Gamma^{(n)}(\mathfrak{h})$, $\|\Psi^n\| = 1$. So clearly $\|\mathrm{d}\Gamma(1)\| = \infty$.

(b) Let $\mathrm{d}\Gamma^{(n)}(b)$ be the restriction of $\mathrm{d}\Gamma(b)$ to $\Gamma^{(n)}(\mathfrak{h})$. Since b is bounded, $\mathrm{d}\Gamma^{(n)}(b)$ is also bounded, in particular (essentially) self-adjoint. Hence

$$(\mathrm{d}\Gamma^{(n)}(b) \pm i)\Gamma^{(n)}(\mathfrak{h}) \tag{13}$$

is dense in $\Gamma^{(n)}(\mathfrak{h})$ (actually equals $\Gamma^{(n)}(\mathfrak{h})$). Now we want to show from (13) that

$$(\mathrm{d}\Gamma(b) \pm i)\Gamma_{\mathrm{fin}}(\mathfrak{h}) \tag{14}$$

is dense: Let $\Psi_{\mathrm{fin}} = \sum_{n\in\mathbb{N}_{\mathrm{fin}}} \Psi^{(n)}$ be an arbitrary element of $\Gamma_{\mathrm{fin}}(\mathfrak{h})$ i.e. $\Psi^{(n)}$ are arbitrary elements of $\Gamma^{(n)}(\mathfrak{h})$ and $\mathbb{N}_{\mathrm{fin}}$ is an arbitrary finite subset of $\mathbb{N}$. We have

$$(\mathrm{d}\Gamma(b) \pm i)\Psi_{\mathrm{fin}} = \sum_{n\in\mathbb{N}_{\mathrm{fin}}} (\mathrm{d}\Gamma(b) \pm i)\Psi^{(n)}. \tag{15}$$

By (13) we can approximate any element of $\Gamma_{\mathrm{fin}}(\mathfrak{h})$ with such vectors, which concludes the proof.

Problem 5. Show that for any bounded s.a. operator b on $\mathfrak{h}$

$$\Gamma(e^{itb}) = e^{it\mathrm{d}\Gamma(b)}. \tag{16}$$

Solution: The r.h.s. is manifestly a group of unitaries with a generator $\mathrm{d}\Gamma(b)$. We check that the l.h.s. is a group of unitaries:

$$\Gamma(e^{it_1 b})\Gamma(e^{it_2 b}) = \Gamma(e^{it_1 b}e^{it_2 b}) = \Gamma(e^{i(t_1+t_2)b}). \tag{17}$$

We compute the generator: For $\Psi \in \Gamma^{(n)}(\mathfrak{h})$ we have

$$\Gamma(e^{itb})\Psi = (e^{itb} \otimes \cdots \otimes e^{itb})\Psi. \tag{18}$$

(Using this formula one can also verify strong continuity of $t \mapsto \Gamma(e^{itb})$). Hence

$$\partial_t|_{t=0}\Gamma(e^{itb})\Psi = \sum_{i=1}^{n}(1 \otimes \cdots b \cdots \otimes 1)\Psi = \mathrm{d}\Gamma(b)\Psi. \tag{19}$$

Hence the two generators coincide on $\Gamma_{\mathrm{fin}}(\mathfrak{h})$ which is the domain of essential self-adjointness of $\mathrm{d}\Gamma(b)$. So their s.a. extensions coincide.

Problem 6. Show that for $\Psi \in \Gamma_{\mathrm{fin}}(\mathfrak{h})$ and $f, g \in L^2(\mathbb{R}^d)$

$$[a(f), a^*(g)]\Psi = \langle f, g\rangle\Psi. \tag{20}$$

Solution: We recall the formulas for creation and annihilation operators:

$$(a(f)\Psi)^{(n)}(k_1, \ldots, k_n) \;=\; \sqrt{n+1}\int d^d p\, \overline{f}(p)\Psi^{(n+1)}(p, k_1, \ldots, k_n), \tag{21}$$

$$(a^*(g)\Psi)^{(n)}(k_1, \ldots, k_n) \;=\; \frac{1}{\sqrt{n}}\sum_{\ell=1}^{n} g(k_\ell)\Psi^{(n-1)}(k_1, \ldots, k_{\ell-1}, k_{\ell+1}, \ldots, k_n). \tag{22}$$

Thus we have

$$(a(f)a^*(g)\Psi)^{(n)}(k_1,\dots,k_n)=\sqrt{n+1}\int d^dp\,\overline{f}(p)(a^*(g)\Psi)^{(n+1)}(p,k_1,\dots,k_n)$$
$$=\int d^dp\,\overline{f}(p)g(p)\Psi^{(n)}(k_1,\dots,k_n)+\sum_{\ell=1}^{n}\int d^dp\,\overline{f}(p)g(k_\ell)\Psi^{(n)}(p,k_1,\dots,k_{\ell-1},k_{\ell+1},\dots,k_n).$$

The sum on the r.h.s. above is cancelled by

$$\begin{aligned}(a^*(g)a(f)\Psi)^{(n)}(k_1,\dots,k_n) &= \frac{1}{\sqrt{n}}\sum_{\ell=1}^{n}g(k_\ell)(a(f)\Psi)^{(n-1)}(k_1,\dots,k_{\ell-1},k_{\ell+1},\dots,k_n)\\ &= \sum_{\ell=1}^{n}g(k_\ell)\overline{f}(p)\Psi^{(n)}(p,k_1,\dots,k_{\ell-1},k_{\ell+1},\dots,k_n).\end{aligned}$$

Problem 7. Let $(\tilde{x},\Lambda)\in\mathcal{P}_+^{\uparrow}$ (proper, ortochronous Poincaré group). Show that the prescription

$$(u_{(\tilde{x},\Lambda)}f)(p)=e^{i\mu(p)t-ipx}\sqrt{\frac{\mu(\Lambda^{-1}p)}{\mu(p)}}f(\Lambda^{-1}p),\quad f\in L^2(\mathbb{R}^d) \tag{23}$$

defines a representation of $\mathcal{P}_+^{\uparrow}$ in unitaries on $L^2(\mathbb{R}^d)$. Here $\tilde{x}=(t,x)$, $\mu(p)=\sqrt{p^2+m^2}$ and $\Lambda^{-1}p$ (action of a Lorentz transformation on a d-vector) is defined by $\Lambda^{-1}(\mu(p),p)=(\mu(\Lambda^{-1}p),\Lambda^{-1}p)$. Hints:

(i) Note that $\frac{d^dp}{\mu(p)}$ is a Lorentz invariant measure i.e.

$$\int\frac{d^dp}{\mu(p)}g(\Lambda^{-1}p)=\int\frac{d^dp}{\mu(p)}g(p). \tag{24}$$

(ii) The multiplication in $\mathcal{P}_+^{\uparrow}$ is defined as follows: $(\tilde{x}_1,\Lambda_1)(\tilde{x}_2,\Lambda_2)=(\tilde{x}_1+\Lambda_1\tilde{x}_2,\Lambda_1\Lambda_2)$

Solution: First, we check that $u_{(\tilde{x},\Lambda)}$ defines an isometry: We have

$$\langle(u_{(\tilde{x},\Lambda)}f),(u_{(\tilde{x},\Lambda)}g)\rangle=\int\frac{d^dp}{\mu(p)}\mu(\Lambda^{-1}p)(\overline{f}\cdot g)(\Lambda^{-1}p)=\int\frac{d^dp}{\mu(p)}\mu(p)(\overline{f}\cdot g)(p)=\langle f,g\rangle. \tag{25}$$

Next, we check multiplicativity

$$\begin{aligned}(u_{(\tilde{x}_1,\Lambda_1)}u_{(\tilde{x}_2,\Lambda_2)}h)(p) &= e^{i\tilde{p}\cdot\tilde{x}_1}\sqrt{\frac{\mu(\Lambda_1^{-1}p)}{\mu(p)}}(u_{(\tilde{x}_2,\Lambda_2)}h)(\Lambda_1^{-1}p)\\ &= e^{i\tilde{p}\cdot\tilde{x}_1+i(\widetilde{\Lambda^{-1}p})\cdot\tilde{x}_2}\sqrt{\frac{\mu(\Lambda_1^{-1}p)}{\mu(p)}}\sqrt{\frac{\mu(\Lambda_2^{-1}\Lambda_1^{-1}p)}{\mu(\Lambda_1^{-1}p)}}h(\Lambda_2^{-1}\Lambda_1^{-1}p)\\ &= e^{i\tilde{p}\cdot\tilde{x}_1+i(\widetilde{\Lambda_1^{-1}p})\cdot\tilde{x}_2}\sqrt{\frac{\mu(\Lambda_2^{-1}\Lambda_1^{-1}p)}{\mu(p)}}h(\Lambda_2^{-1}\Lambda_1^{-1}p)\\ &= e^{i\tilde{p}\cdot\tilde{x}_1+i\tilde{p}\cdot\Lambda_1\tilde{x}_2}\sqrt{\frac{\mu(\Lambda_2^{-1}\Lambda_1^{-1}p)}{\mu(p)}}h(\Lambda_2^{-1}\Lambda_1^{-1}p)\\ &= (u_{(\tilde{x}_1+\Lambda_1\tilde{x}_2,\Lambda_1\Lambda_2)}h)(p),\end{aligned} \tag{26}$$

where in the next to the last step we made use of $(\mu(\Lambda^{-1}p), \Lambda^{-1}p) = \Lambda^{-1}(\mu(p), p)$ and $\Lambda^{-1}\tilde{p} \cdot \tilde{x} = \tilde{p} \cdot \Lambda\tilde{x}$. (Here we use that $(\Lambda\tilde{p}) \cdot (\Lambda\tilde{x}) = \tilde{p} \cdot \tilde{x}$ for all $\tilde{p}$, $\tilde{x}$, which is equivalent to $g = \Lambda^T g \Lambda$, which is equivalent to $g = \Lambda g \Lambda^T$.) From this we get, in particular that $u_{(x,\Lambda)}$ is invertible because $u_{(0,I)} = \mathrm{id}$ and $\mathcal{P}_+^\uparrow$ is a group. Hence it is unitary. (Note that $(x, \Lambda)(-\Lambda^{-1}x, \Lambda^{-1}) = (-\Lambda^{-1}x, \Lambda^{-1})(x, \Lambda) = (0, I)$).

To be discussed in class: 8.6.2017