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Algebraic Quantum Field Theory
Homework Sheet 3

Problem 1. We define $\mathcal{V}$ as the polynomial $*$-algebra generated by $V(z)$, $(z \in \mathbb{C}^n)$, satisfying the relations $V(z)^* = V(-z)$ and

$$V(z)V(z') = e^{i\mathrm{Im}\langle z,z'\rangle} V(z')V(z). \tag{1}$$

Show that given any representation π of $\mathcal{W}$, we can use the prescription $\pi_1(V(z)) := \pi(W(z))$ to obtain a well-defined representation of $\mathcal{V}$.

Problem 2. Let $R(\lambda, z)$ be as in the definition of the pre-resolvent algebra. Show that in any $*$-representation π

$$\|\pi(R(\lambda, z))\| \leq \frac{1}{\lambda}, \tag{2}$$

and therefore the norm of any element of the pre-resolvent algebra is bounded.

Problem 3. Let $\mathcal{A} \subset B(\mathcal{H})$ be a norm-closed unital $*$-algebra of operators (i.e. a concrete C^*-algebra) which acts irreducibly on a separable Hilbert space $\mathcal{H}$ (i.e. $\mathcal{A}' = \mathbb{C}I$). Suppose that $\mathcal{A}$ contains one non-zero compact operator C. Show that then it contains all of them. Hints:

(i) Note that you can assume that $C = C^*$.

(ii) Recall that the spectrum of a self-adjoint compact operator is discrete and has at most one accumulation point (at zero). Non-zero eigenvalues have finite multiplicity.

(iii) If A is a self-adjoint element of a C^*-algebra then also $f(A)$ is an element of this C^*-algebra for any continuous function f.

(iv) Exploiting the above information show that $\mathcal{A}$ contains a finite-dimensional projection. Using the bicommutant theorem and irreducibility conclude that it contains a one-dimensional projection.

(v) Using irreducibility again show that it contains all compact operators.

Problem 4. (a) Show that $N := \mathrm{d}\Gamma(1)$ is an unbounded operator on $\Gamma(\mathfrak{h})$.
(b) Show that for any bounded s.a. operator b on $\mathfrak{h}$ its second quantization $\mathrm{d}\Gamma(b)$ is essentially self adjoint on $\Gamma_{\mathrm{fin}}(\mathfrak{h})$. Hints:

(i) Recall that a symmetric operator T on a domain $D(T)$ is essentially self adjoint if it has a unique self-adjoint extension.

(ii) Essential self-adjointness is equivalent to $(T \pm i)D(T)$ being dense.

Problem 5. Show that for any bounded s.a. operator b on $\mathfrak{h}$

$$\Gamma(e^{itb}) = e^{it\mathrm{d}\Gamma(b)}. \tag{3}$$

Problem 6. Show that for $\Psi \in \Gamma_{\mathrm{fin}}(\mathfrak{h})$ and $f, g \in L^2(\mathbb{R}^d)$

$$[a(f), a^*(g)]\Psi = \langle f, g\rangle \Psi. \tag{4}$$

Problem 7. Let $(\tilde{x}, \Lambda) \in \mathcal{P}_+^\uparrow$ (proper, ortochronous Poincaré group). Show that the prescription

$$(u_{(\tilde{x},\Lambda)}f)(p) = e^{i\mu(p)t - ipx}\sqrt{\frac{\mu(\Lambda^{-1}p)}{\mu(p)}} f(\Lambda^{-1}p), \quad f \in L^2(\mathbb{R}^d) \tag{5}$$

defines a representation of $\mathcal{P}_+^\uparrow$ in unitaries on $L^2(\mathbb{R}^d)$. Here $\tilde{x} = (t, x)$, $\mu(p) = \sqrt{p^2 + m^2}$ and $\Lambda^{-1}p$ (action of a Lorentz transformation on a d-vector) is defined by $\Lambda^{-1}(\mu(p), p) = (\mu(\Lambda^{-1}p), \Lambda^{-1}p)$. Hints:

(i) Note that $\frac{d^dp}{\mu(p)}$ is a Lorentz invariant measure i.e.

$$\int \frac{d^dp}{\mu(p)} g(\Lambda^{-1}p) = \int \frac{d^dp}{\mu(p)} g(p). \tag{6}$$

(ii) The multiplication in $\mathcal{P}_+^\uparrow$ is defined as follows: $(\tilde{x}_1, \Lambda_1)(\tilde{x}_2, \Lambda_2) = (\tilde{x}_1 + \Lambda_1\tilde{x}_2, \Lambda_1\Lambda_2)$

To be discussed in class: 8.6.2017