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LMU and TU München
Sommersemester 2017

Algebraic Quantum Field Theory
Homework Sheet 4

Problem 1. Show that in the sense of quadratic forms on $D \times D$, where

$$D = \{\, \Psi \in \Gamma_{\mathrm{fin}}(\mathfrak{h}) \mid \Psi^{(n)} \in S(\mathbb{R}^{nd}) \text{ for all } n \,\}, \tag{1}$$

we have the following representations for the (free) Hamiltonian $H := \mathrm{d}\Gamma(\mu_m(p))$:

$$\begin{aligned} \mathrm{d}\Gamma(\mu_m(p)) &= \int d^dk\, \mu_m(k) a^*(k) a(k) \\ &= \frac{1}{2} \int d^dx \left(:\pi_{\mu_m}^2(x): + :\nabla\varphi_{\mu_m}^2(x): + m^2 :\varphi_{\mu_m}^2(x): \right). \end{aligned} \tag{2}$$

Here $\mu_m(p) = \sqrt{p^2 + m^2}$ and

$$\varphi_{\mu_m}(x) := \frac{1}{(2\pi)^{d/2}} \int \frac{d^dk}{\sqrt{2\mu_m(k)}} \big(e^{-ikx} a^*(k) + e^{ikx} a(k) \big), \tag{3}$$

$$\pi_{\mu_m}(x) := \frac{i}{(2\pi)^{d/2}} \int d^dk \sqrt{\frac{\mu_m(k)}{2}} \big(e^{-ikx} a^*(k) - e^{ikx} a(k) \big). \tag{4}$$

The Wick ordering $:(\cdots):$ means shifting creation operators to the left and annihilation operators to the right, ignoring the commutators. For example

$$: (a^*(k_1)a^*(k_2) + a^*(k_1)a(k_2) + a(k_1)a^*(k_2) + a(k_1)a(k_2)) : \tag{5}$$

$$= a^*(k_1)a^*(k_2) + a^*(k_1)a(k_2) + a^*(k_2)a(k_1) + a(k_1)a(k_2). \tag{6}$$

Problem 2. The interaction Hamiltonian

$$H_I(g) := \lambda \int_{\mathbb{R}^d} d^dx\, g(x) :\varphi_{\mu_m}(x)^4:, \quad g \in C_0^\infty(\mathbb{R}^d), \quad \lambda > 0, \tag{7}$$

is well defined as a quadratic form on $D \times D$ (this can be taken for granted). Show that this quadratic form cannot arise for $d > 1$ from an operator which has Ω in its domain. Hint: Consider the formal expression for $H_I(g)\Omega$ which is of the form $(0, 0, 0, 0, \psi^{(4)}, 0, \ldots)$ and show that $\psi^{(4)}$ is not square integrable.

Problem 3. Let $\mathcal{D} = S(\mathbb{R}^d)$, $d = 3$, be the symplectic space with the standard symplectic form. Consider the representations of $\mathcal{W}$ on Fock space given by

$$\rho_{\mu_m}(W(f)) = e^{i(\varphi_{\mu_m}(\mathrm{Re}\, f) + \pi_{\mu_m}(\mathrm{Im}\, f))}. \tag{8}$$

These representations are irreducible (this can be taken for granted). Show that $\rho_{\mu_{m_1}}$ is not unitarily equivalent to $\rho_{\mu_{m_2}}$ if $m_1 \neq m_2$, $m_1, m_2 > 0$. Hints:

(i) Suppose, by contradiction, that there is a unitary T on the Fock space which intertwines the two representations. Let $E \ni (a, R) \to U(a, R)$ be the unitary representation of the group of Euclidean motions in the $t = 0$ plane (space translations and rotations) which implements the corresponding automorphisms in the two representations. Show that $C(a, R) := T^{-1}U(a, R)^*TU(a, R)$ must be a multiple of the identity.

(ii) Use that E has no non-trivial one-dimensional representations.

(iii) Use that multiples of Ω are the only vectors in Fock space invariant under $(a, R) \to U(a, R)$.

To be discussed in class: 29.06.2017