Prof. UAM dr hab. Wojciech Dybalski

AMU Poznań

Winter semester 2025/26

Stochastic Differential Equations
Homework Sheet 5 - solutions

Problem 1. Let $\{B_t\}_{t\in\mathbb{R}_+}$ be a one-dimensional Brownian motion starting at $x \in \mathbb{R}$ and let $c > 0$ be a constant. Prove that the process

$$\tilde{B}_t := \frac{1}{c} B_{c^2 t} \tag{1}$$

is also a Brownian motion. Hint: Show that the finite-dimensional distributions of $\{B_t\}_{t\in\mathbb{R}_+}$ and $\{\tilde{B}_t\}_{t\in\mathbb{R}_+}$ coincide.

Solution. We have

$$\begin{aligned}
& P^x(\tilde{B}_{t_1} \in F_1, \cdots, \tilde{B}_{t_k} \in F_k) \\
= \; & P^x(\tilde{B}_{c^2 t_1} \in cF_1, \cdots, \tilde{B}_{c^2 t_k} \in cF_k) \\
= \; & \int_{cF_1\times\cdots\times cF_k} p(c^2 t_1, x, x_1)\cdots p(c^2(t_{k-1}-t_{k-2}), x_{k-2}, x_{k-1}) p(c^2(t_k - t_{k-1}), x_{k-1}, x_k) dx_1 \cdots dx_k \\
= \; & \int_{F_1\times\cdots\times F_k} cp(c^2 t_1, c\tilde{x}, c\tilde{x}_1)\cdots cp(c^2(t_{k-1}-t_{k-2}), c\tilde{x}_{k-2}, c\tilde{x}_{k-1}) p(c^2(t_k - t_{k-1}), c\tilde{x}_{k-1}, c\tilde{x}_k) d\tilde{x}_1 \cdots d\tilde{x}_k \\
= \; & \int_{F_1\times\cdots\times F_k} p(t_1, \tilde{x}, \tilde{x}_1)\cdots p((t_{k-1}-t_{k-2}), \tilde{x}_{k-2}, \tilde{x}_{k-1}) p((t_k - t_{k-1}), \tilde{x}_{k-1}, \tilde{x}_k) d\tilde{x}_1 \cdots d\tilde{x}_k \qquad (2) \\
= \; & P^x(B_{t_1} \in F_1, \cdots, B_{t_k} \in F_k).
\end{aligned}$$

We used in (2) that

$$cp(c^2 t, cx, cy) = c(2\pi t c^2)^{-1/2} \exp\big(-\frac{c^2|x-y|^2}{2c^2 t}\big) = p(t, x, y). \tag{3}$$

Side remark: In other words, $B_t \stackrel{d}{=} \frac{1}{c} B_{c^2 t}$, where $\stackrel{d}{=}$ means equality in distribution. This (statistical) self-similarity property is reminiscent of fractals - objects which retain their essential features under zooming (think of a coastal line). In a future Homework Sheet we will encounter the Cantor set which is the simplest example of a deterministic (i.e. non-statistical) fractal.

Problem 2. Show that the function

$$g(s) = \begin{cases} s\sin\left(\frac{1}{s}\right), & s \neq 0, \\ 0, & s = 0. \end{cases} \tag{4}$$

has infinite total variation on $[0, 1]$.

Solution. Fix $n \in \mathbb{N}$ and set $I_n = [s_{n+1}, s_n]$ with $s_n := 1/(n\pi)$. Note that

$$u_n := n\pi + \tfrac{\pi}{2} \in [n\pi, (n+1)\pi],$$

so the point

$$r_n := \frac{1}{u_n} = \frac{1}{n\pi + \frac{\pi}{2}}$$

lies in I_n. Since $\sin(u_n) = \sin\big(n\pi + \frac{\pi}{2}\big) = (-1)^n$, we have

$$g(r_n) = r_n \sin\big(\tfrac{1}{r_n}\big) = r_n \sin(u_n) = (-1)^n r_n.$$

Also $g(s_{n+1}) = g(s_n) = 0$ because $1/s_{n+1} = (n+1)\pi$ and $1/s_n = n\pi$ are zeros of sin.

Consider the partition $\{[s_{n+1}, r_n] \cup [r_n, s_n]\}$ of $I_n = [s_{n+1}, s_n]$. By the definition of variation as the supremum over partitions,

$$\begin{aligned} \mathrm{V}_{I_n}(g) &\geq |g(r_n) - g(s_{n+1})| + |g(s_n) - g(r_n)| \\ &= |(-1)^n r_n - 0| + |0 - (-1)^n r_n| = r_n + r_n = 2\, r_n \\ &= 2\left(\frac{1}{n\pi + \frac{\pi}{2}}\right). \end{aligned}$$

We observe that $V_0^1(g) \geq \sum_{n=1}^{\infty} V_{I_n}(g)$ (in fact, if we restrict the supremum in $V_0^1(g)$ to partitions containing the end-points of intervals I_n, the quantity will get smaller. But this restriction does not affect the sets of partitions inside each I_n).

Summing over n yields

$$V_0^1(g) \;\geq\; \sum_{n=1}^{\infty} 2\left(\frac{1}{n\pi + \frac{\pi}{2}}\right),$$

which diverges by comparison with the harmonic series (since $n\pi + \frac{\pi}{2} \leq (n+1)\pi$ gives $\frac{1}{n\pi + \frac{\pi}{2}} \geq \frac{1}{(n+1)\pi}$). Thus the total variation of g on $[0,1]$ is infinite.

It is a classical fact that the series $\sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges. This can be seen as follows: Clearly

$$\int_{n+1}^{n+2} \frac{1}{x} dx \leq \frac{1}{n+1}. \tag{5}$$

Therefore

$$\sum_{n=1}^{N} \int_{n+1}^{n+2} \frac{1}{x} dx = \sum_{n=1}^{N} [\log(n+2) - \log(n+1)] = \log(N+2) - \log(2) \leq \sum_{n=1}^{N} \frac{1}{n+1}. \tag{6}$$

But $\lim_{N\to\infty} \log(N+2) = \infty$.

Problem 3. Suppose that $g \in C^1(\mathbb{R})$. Show that $V_a^b(g) = \int_a^b |g'(s)| ds$.

Hint: Consider separately the inequalities $V_a^b(g) \leq \int_a^b |g'(s)| ds$ and $V_a^b(g) \geq \int_a^b |g'(s)| ds$.

Solution. Step 1. Upper bound. For any partition $\Pi = \{a = s_0 < s_1 < \cdots < s_n = b\}$,

$$f(s_i) - f(s_{i-1}) = \int_{s_{i-1}}^{s_i} f'(s)\,ds. \tag{7}$$

Hence

$$\begin{aligned}\sum_{i=1}^{n} |f(s_i) - f(s_{i-1})| &= \sum_{i=1}^{n} \left| \int_{s_{i-1}}^{s_i} f'(s)\,ds \right| \\ &\leq \sum_{i=1}^{n} \int_{s_{i-1}}^{s_i} |f'(s)|\,ds = \int_a^b |f'(s)|\,ds. \end{aligned} \tag{8}$$

Taking the supremum over all partitions gives

$$V_a^b(f) \leq \int_a^b |f'(s)|\,ds. \tag{9}$$

Step 2. Lower bound. Because f' is continuous, so is $|f'|$, hence Riemann integrable. Given $\varepsilon > 0$, there exists a tagged partition (Π, ξ) with mesh $|\Pi|$ small enough such that

$$\int_a^b |f'(s)|\,ds \leq \sum_{i=1}^{n} |f'(\xi_i)|(s_i - s_{i-1}) + \varepsilon. \tag{10}$$

By the mean value theorem, for each i there exists $\eta_i \in (s_{i-1}, s_i)$ such that

$$f(s_i) - f(s_{i-1}) = f'(\eta_i)\,(s_i - s_{i-1}). \tag{11}$$

Therefore,

$$\sum_{i=1}^{n} |f(s_i) - f(s_{i-1})| = \sum_{i=1}^{n} |f'(\eta_i)|(s_i - s_{i-1}). \tag{12}$$

Since the Riemann integral does not depend on the tags of the partition, we can set $\eta_i = \xi_i$. Combining (10), (12) we get

$$\sum_{i=1}^{n} |f(s_i) - f(s_{i-1})| \geq \int_a^b |f'(s)|\,ds - \varepsilon. \tag{13}$$

Since $\varepsilon > 0$ is arbitrary and $V_a^b(f)$ is the supremum over all partitions,

$$V_a^b(f) \geq \int_a^b |f'(s)|\,ds. \tag{14}$$

To be discussed in class: 13.11.2025