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# Algebraic Quantum Field Theory
# Homework Sheet 5

**Problem 1.** Show that for $\tilde{p}_1, \tilde{p}_2 \in H_m$, where $H_m$ is the mass hyperboloid, we have that $\tilde{p} := \tilde{p}_1 - \tilde{p}_2$ is a spacelike vector, if non-zero. (Recall that $\tilde{p} = (p^0, p)$ is spacelike if $(p^0)^2 < p^2$).

**Problem 2.** Show that the subspace of functions

$$D = \mathrm{Span}\,\{(p_1, p_2) \mapsto g_1(p_1) g_2(p_2) \,|\, g_1, g_2 \in C_0^\infty(\mathbb{R}) \;\mathrm{supp}\, g_1 \cap \mathrm{supp}\, g_2 = \emptyset\,\} \tag{1}$$

is dense in $L^2(\mathbb{R} \times \mathbb{R})$.

**Problem 3.** Consider a regular, positive energy solution of the Klein Gordon equation:

$$g_t(x) := (2\pi)^{-\frac{d}{2}} \int d^d p \, e^{-i\mu_m(p)t + ipx} \widehat{g}(p), \quad \widehat{g} \in C_0^\infty(\mathbb{R}^d), \tag{2}$$

whose velocity support is given by

$$V(g) := \{\, \nabla \mu_m(p) \,|\, p \in \mathrm{supp}\, \widehat{g}\,\}. \tag{3}$$

Let $\chi_+ \in C_0^\infty(\mathbb{R}^d)$ be equal to one on $V(g)$ and vanish on a complement of a slightly larger set. Let $\chi_- := 1 - \chi_+$. Show that

$$|\chi_-(x/t) g_t(x)| \le \frac{c}{|t|}. \tag{4}$$

Hint: set $\Omega(p) = (-\mu_m(p) + px/t)$ and note the relation

$$L e^{it\Omega(p)} = e^{it\Omega(p)}, \tag{5}$$

where $L = \frac{1}{it} \frac{\nabla\Omega(p) \cdot \nabla}{|\nabla\Omega(p)|^2}$

**Problem 4.** Consider an abstract Haag-Kastler net of von Neumann algebras. Let $B \in \mathcal{A}$, $f \in S(\mathbb{R}^{d+1})$. Show that

$$B(f) = \int dt dx \, f(t,x) B(t,x), \tag{6}$$

defined as a weak integral (i.e. in the sense of matrix elements) is an element of $\mathcal{A}$. Hint: Consider first $B \in \mathcal{A}(\mathcal{O})$ for some open, bounded $\mathcal{O}$ and $f \in C_0^\infty(\mathbb{R}^{d+1})$. Use the bicommutant theorem to show that $B(f) \in \mathcal{A}(\mathcal{O}_1)$ for some larger but still bounded open subset $\mathcal{O}_1$.

**To be discussed in class:** 20.07.2017