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# Stochastic Differential Equations
# Homework Sheet 6 - solutions

**Problem 1.** Consider $S_n := \sum_{k=1}^{2^n} \left( B_{kT/2^n} - B_{(k-1)T/2^n} \right)^2$ like in class. Show that

$$S_n \underset{n\to\infty}{\to} T \text{ in } L^2(\Omega, P). \tag{1}$$

Hints: Justify that it suffices to show $\mathrm{var}(S_n) \underset{n\to\infty}{\to} 0$. Relate the latter variance to $\mathrm{var}((\Delta B_k)^2)$ using HS2, Problem 2. Compute $\mathrm{var}((\Delta B_k)^2)$ with the help of HS3, Problem 1.

**Solution.** We have $S_n := \sum_{k=1}^{2^n} \left( B_{kT/2^n} - B_{(k-1)T/2^n} \right)^2 =: \sum_k (\Delta B_k)^2$. We note that

$$E[(\Delta B_k)^2] = 2^{-n}T, \tag{2}$$
$$\mathrm{var}((\Delta B_k)^2) = E[(\Delta B_k)^4] - E[(\Delta B_k)^2]^2 = 3\left(2^{-n}T\right)^2 - (2^{-n}T)^2 = 2\left(2^{-n}T\right)^2, \tag{3}$$

where we used that for $Z \sim N(0, \sigma^2)$ there holds $E(Z^4) = 3\sigma^4$ (HS3, Problem 1). We have

$$E[S_n] = \sum_{k=1}^{2^n} E[(\Delta B_k)^2] = T. \tag{4}$$

Since the increments are independent, we have, (HS2, Problem 2),

$$E[(S_n - E[S_n])^2] = \mathrm{var}(S_n) = \sum_{k=1}^{2^n} \mathrm{var}((\Delta B_k)^2) = \sum_{k=1}^{2^n} 2\left(2^{-n}T\right)^2 = 2^n 2\left(2^{-n}T\right)^2 \to 0. \tag{5}$$

Considering (4) we conclude the proof.

**Problem 2.** Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let

$$\mathcal{N} = \{\, N \subset \Omega \,:\, \exists Z \in \mathcal{F} \text{ with } N \subset Z, P(Z) = 0 \,\}. \tag{6}$$

Define the completion by

$$\mathcal{F}^* = \{\, A \cup N \,:\, A \in \mathcal{F},\, N \in \mathcal{N} \,\}, \quad P^*(A \cup N) = P(A). \tag{7}$$

Show that $\mathcal{F}^*$ is a $\sigma$-field and $P^*$ is a well defined measure in $\mathcal{F}^*$. (Note that $\mathcal{F} \subset \mathcal{F}^*$, since $A = A \cup \emptyset$, $\emptyset \in \mathcal{N}$).

**Solution.** Let us show that $\mathcal{F}^*$ is a $\sigma$-field:

(i) Clearly $\emptyset \in \mathcal{F}$ since it is a set of measure zero.

(ii) Suppose $A \cup N \in \mathcal{F}^*$ and $N \subset Z$, $Z \in \mathcal{F}$, $P(Z) = 0$. Then (by picture)

$$(A \cup N)^c = A^c \cap N^c = A^c \cap (Z^c \cup Z \backslash N) = (A^c \cap Z^c) \cup (A^c \cap (Z \backslash N)). \tag{8}$$

We have $A^c \cap Z^c \in \mathcal{F}$ and $A^c \cap (Z \backslash N) \subset Z$, where $P(Z) = 0$.

(iii) Suppose $A_i \cup N_i \in \mathcal{F}^*$. Then

$$\bigcup_{i=1}^{\infty}(A_i \cup N_i) = (\bigcup_{i=1}^{\infty} A_i) \cup (\bigcup_{i=1}^{\infty} N_i) \tag{9}$$

We have $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ and $\bigcup_{i=1}^{\infty} N_i \subset \bigcup_{i=1}^{\infty} Z_i$, where the latter set is of $P$-measure zero.

Let us now check that the map $P^*$ is well defined. Suppose $A_1 \cup N_1 = A_2 \cup N_2$. Then $A_1 \subset A_2 \cup N_2 \subset A_2 \cup Z_2$ and $A_2 \subset A_1 \cup N_1 \subset A_1 \cup Z_1$, hence

$$P(A_1) \leq P(A_2 \cup Z_2) \leq P(A_2) + P(Z_2) = P(A_2), \tag{10}$$

$$P(A_2) \leq P(A_1 \cup Z_1) \leq P(A_1) + P(Z_1) = P(A_1), \tag{11}$$

hence $P(A_1) = P(A_2)$, thus

$$P^*(A_1 \cup N_1) = P(A_1) = P(A_2) = P^*(A_2 \cup N_2). \tag{12}$$

Now we check the properties of the measure:

(a) $P^*(\emptyset) = P(\emptyset) = 0$, $P^*(\Omega) = P(\Omega) = 1$

(b) If $\{A_i \cup N_i\}_{i\in\mathbb{N}}$ are disjoint, then $\{A_i\}_{i\in\mathbb{N}}$ are disjoint and $\{N_i\}_{i\in\mathbb{N}}$ are disjoint. Hence

$$P^*(\bigsqcup_{i\in\mathbb{N}}(A_i \cup N_i)) = P^*((\bigsqcup_{i\in\mathbb{N}} A_i) \cup (\bigsqcup_{i\in\mathbb{N}} N_i)) = P(\bigsqcup_{i\in\mathbb{N}} A_i) = \sum_{i\in\mathbb{N}} P(A_i) = \sum_{i\in\mathbb{N}} P^*(A_i \cup N_i)$$

**Problem 3.** Define the *middle–third Cantor set* $C \subset [0, 1]$ as follows:

(i) Start with the closed interval

$$C_0 = [0, 1].$$

(ii) Given $C_n$, obtain $C_{n+1}$ by removing from each closed interval of $C_n$ its open middle third. For example:

$$C_1 = [0, 1/3] \cup [2/3, 1], \tag{13}$$

$$C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]. \tag{14}$$

Continuing indefinitely produces a decreasing sequence of closed sets

$$C_0 \supset C_1 \supset C_2 \supset \cdots$$

where $C_{n+1}$ is obtained from $C_n$ by removing the open middle third of each interval.

(iii) Define

$$C = \bigcap_{n=0}^{\infty} C_n.$$

This set $C$ is called the *Cantor set*.

Prove that the Lebesgue measure of $C$ is zero, i.e. $\mu(C) = 0$.

**Solution.** At the first step, the total length of $C_0 = [0,1]$ is 1. We remove the open interval (1/3, 2/3) of length 1/3, so the total length of $C_1$ is

$$\mu(C_1) = 1 - \frac{1}{3} = \frac{2}{3}.$$

At the second step, each of the two intervals in $C_1$ has length 1/3. Removing the middle third from each means removing two open intervals, each of length 1/9, so the remaining total length is

$$\mu(C_2) = \frac{2}{3} - \frac{2}{9} = \frac{4}{9} = \left(\frac{2}{3}\right)^2.$$

In general, after $n$ steps, $C_n$ consists of $2^n$ closed intervals, each of length $3^{-n}$. Hence

$$\mu(C_n) = 2^n \cdot 3^{-n} = \left(\frac{2}{3}\right)^n.$$

Since the sequence $(C_n)_{n\in\mathbb{N}}$ is decreasing and $C = \bigcap_{n=0}^{\infty} C_n$, continuity of the measure from above (HS1, Problem 6) gives

$$\mu(C) = \lim_{n\to\infty} \mu(C_n) = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0.$$

Therefore, the Cantor set $C$ has Lebesgue measure zero.

**Problem 4.** Let $C \subset [0,1]$ be the middle-third Cantor set as above. Consider the space of infinite binary sequences

$$2^{\mathbb{N}} = \{(b_k)_{k\geq 1} : b_k \in \{0,1\}\}.$$

Define

$$\Phi : 2^{\mathbb{N}} \to [0,1], \qquad \Phi\big((b_k)_{k\geq 1}\big) = \sum_{k=1}^{\infty} \frac{2b_k}{3^k}. \tag{15}$$

Note that $\Phi(2^{\mathbb{N}}) \subset C$. In fact, every number $x \in [0,1]$ can be written in base 3 (ternary) expansion as $x = 0.a_1a_2a_3\ldots_3 = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$ where $a_k \in \{0,1,2\}$. For example,

$$0 = 0.0000\ldots_3 \tag{16}$$
$$1 = 0.2222\ldots_3 \tag{17}$$
$$\frac{1}{3} = 0.1000\ldots_3 \tag{18}$$
$$\frac{2}{3} = 0.2000\ldots_3 \tag{19}$$

Therefore, removing the middle third $(\frac{1}{3}, \frac{2}{3})$ to build $C_1$, as in (13) above, means removing all digits whose first ternary digit is 1:

$$(\frac{1}{3}, \frac{2}{3}) = \{ x \in [0,1] \,:\, a_1 = 1 \}. \tag{20}$$

Hence, $C_1 = \{ x \in [0,1] \,:\, a_1 \in \{0,2\} \}$. Analogously, one can see that $C_n$ consists of numbers whose first $n$ ternary digits belong to $\{0,2\}$, hence $C$ consists of numbers whose all ternary digits belong to $\{0,2\}$. Such numbers occur on the r.h.s. of (15).

(a) Show that for every $x \in C$ there exists $(b_k)_{k\geq 1} \in 2^{\mathbb{N}}$ with $x = \Phi((b_k)_{k\geq 1})$, i.e. $\Phi$ is surjective.

(b) Prove that $\Phi$ is injective.

(c) Deduce that $|C| = |2^{\mathbb{N}}| = \mathfrak{c}$; i.e., the Cantor set has the cardinality of the continuum.

**Solution.** (a) Every $x \in C$ has a ternary expansion $x = \sum_{k\geq 1} \frac{a_k}{3^k}$ with $a_k \in \{0,2\}$. Setting $b_k = a_k/2 \in \{0,1\}$ gives $x = \Phi((b_k)_{k\geq 1})$.

(b) Suppose, by contradiction, that two different binary sequences $(b_k)_{k\in\mathbb{N}}$ and $(c_k)_{k\geq 1}$ gave the same real number: $\Phi((b_k)_{k\geq 1}) = \Phi((c_k)_{k\geq 1})$. Then the corresponding ternary digit sequences

$$a_k = 2b_k, \quad d_k = 2c_k \tag{21}$$

would give two different ternary expansions using only digits 0 and 2 for the same real number. But in base-3 the only way a number can have two expansions is:

$$\ldots x_n 000 \ldots_3 = \ldots (x_n - 1)222 \ldots_3\,, \quad x_n \in \{1,2\}. \tag{22}$$

Thus, only one of the two numbers can have all digits in $\{0,1\}$.

(c) From (a)–(b), $\Phi$ is a bijection $2^{\mathbb{N}} \to C$. Hence, $|C| = |2^{\mathbb{N}}| = 2^{\aleph_0} = \mathfrak{c}$.

**To be discussed in class:** 21.11.2025