Prof. UAM dr hab. Wojciech Dybalski AMU Poznań
Winter semester 2025/26

Stochastic Differential Equations
Homework Sheet 6

Problem 1. Consider $S_n := \sum_{k=1}^{2^n} \left(B_{kT/2^n} - B_{(k-1)T/2^n}\right)^2$ like in class. Show that

$$S_n \underset{n\to\infty}{\to} T \text{ in } L^2(\Omega, P). \tag{1}$$

Hints: Justify that it suffices to show $\mathrm{var}(S_n) \underset{n\to\infty}{\to} 0$. Relate the latter variance to $\mathrm{var}((\Delta B_k)^2)$ using HS2, Problem 2. Compute $\mathrm{var}((\Delta B_k)^2)$ with the help of HS3, Problem 1.

Problem 2. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let

$$\mathcal{N} = \{\, N \subset \Omega \,:\, \exists Z \in \mathcal{F} \text{ with } N \subset Z, P(Z) = 0 \,\}. \tag{2}$$

Define the completion by

$$\mathcal{F}^* = \{\, A \cup N \,:\, A \in \mathcal{F},\, N \in \mathcal{N} \,\}, \quad P^*(A \cup N) = P(A). \tag{3}$$

Show that $\mathcal{F}^*$ is a σ-field and P^* is a well defined measure in $\mathcal{F}^*$. (Note that $\mathcal{F} \subset \mathcal{F}^*$, since $A = A \cup \emptyset$, $\emptyset \in \mathcal{N}$).

Problem 3. Define the *middle–third Cantor set* $C \subset [0, 1]$ as follows:

(i) Start with the closed interval
$$C_0 = [0, 1].$$

(ii) Given C_n, obtain C_{n+1} by removing from each closed interval of C_n its open middle third. For example:

$$C_1 = [0, 1/3] \cup [2/3, 1], \tag{4}$$
$$C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]. \tag{5}$$

Continuing indefinitely produces a decreasing sequence of closed sets

$$C_0 \supset C_1 \supset C_2 \supset \cdots$$

where C_{n+1} is obtained from C_n by removing the open middle third of each interval.

(iii) Define

$$C = \bigcap_{n=0}^{\infty} C_n.$$

This set C is called the *Cantor set.*

Prove that the Lebesgue measure of C is zero, i.e. $\mu(C) = 0$.

Problem 4. Let $C \subset [0,1]$ be the middle-third Cantor set as above. Consider the space of infinite binary sequences

$$2^{\mathbb{N}} = \{(b_k)_{k\geq 1} : b_k \in \{0,1\}\}.$$

Define

$$\Phi : 2^{\mathbb{N}} \to [0,1], \qquad \Phi\big((b_k)_{k\geq 1}\big) = \sum_{k=1}^{\infty} \frac{2b_k}{3^k}. \tag{6}$$

Note that $\Phi(2^{\mathbb{N}}) \subset C$. In fact, every number $x \in [0,1]$ can be written in base 3 (ternary) expansion as $x = 0.a_1a_2a_3\ldots_3 = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$ where $a_k \in \{0,1,2\}$. For example,

$$0 = 0.0000\ldots_3 \tag{7}$$

$$1 = 0.2222\ldots_3 \tag{8}$$

$$\frac{1}{3} = 0.1000\ldots_3 \tag{9}$$

$$\frac{2}{3} = 0.2000\ldots_3 \tag{10}$$

Therefore, removing the middle third $(\frac{1}{3}, \frac{2}{3})$ to build C_1, as in (4) above, means removing all digits whose first ternary digit is 1:

$$(\frac{1}{3}, \frac{2}{3}) = \{\, x \in [0,1] \;:\; a_1 = 1 \,\}. \tag{11}$$

Hence, $C_1 = \{\, x \in [0,1] \;:\; a_1 \in \{0,2\} \,\}$. Analogously, one can see that C_n consists of numbers whose first n ternary digits belong to $\{0,2\}$, hence C consists of numbers whose all ternary digits belong to $\{0,2\}$. Such numbers occur on the r.h.s. of (6).

(a) Show that for every $x \in C$ there exists $(b_k)_{k\geq 1} \in 2^{\mathbb{N}}$ with $x = \Phi((b_k)_{k\geq 1})$, i.e. Φ is surjective.

(b) Prove that Φ is injective.

(c) Deduce that $|C| = |2^{\mathbb{N}}| = \mathfrak{c}$; i.e., the Cantor set has the cardinality of the continuum.

To be discussed in class: 21.11.2025