Prof. UAM dr hab. Wojciech Dybalski

AMU Poznań
Winter semester 2025/26

Stochastic Differential Equations
Homework Sheet 7 - solutions

Problem 1. Let $h \in L^2(\mathbb{R})$. Show that

$$\lim_{s\to 0} \|h(\cdot + s) - h(\,\cdot\,)\|^2_{L^2(\mathbb{R})} = \lim_{s\to 0} \int |h(t+s) - h(t)|^2 dt = 0. \tag{1}$$

Hint: One way to solve this problem is to use the Plancherel theorem: For $f \in L^2(\mathbb{R})$, the Fourier transform $\hat{f}(k) = \int e^{ikt} f(t) dt$ [1] is also in $L^2(\mathbb{R})$ and satisfies

$$\|f\|_{L^2(\mathbb{R})} = \frac{1}{\sqrt{2\pi}} \|\hat{f}\|_{L^2(\mathbb{R})}. \tag{2}$$

Solution. By the Plancherel theorem and dominated convergence

$$\lim_{s'\to 0} \int dt\, |h_1(t+s', \omega) - h_1(t, \omega)|^2 = \lim_{s'\to 0} \frac{1}{\sqrt{2\pi}} \int |e^{-iks'} - 1|^2 |\hat{h}_1(k, \omega)|^2 dk = 0. \tag{3}$$

Side remark: This is a special case of the Stone theorem: Let D be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Then $s \mapsto e^{isD}$ is a strongly continuous group of unitaries, i.e. for any $h \in \mathcal{H}$,

$$\lim_{s\to 0} \|e^{isD} h - h\| = 0. \tag{4}$$

In our case $\mathcal{H} = L^2(\mathbb{R})$ and $D = \frac{1}{i}\frac{d}{dt}$ is the generator of translations (i.e. the quantum-mechanical momentum operator). The identity $(e^{isD} h)(t) = h(t+s)$ is plausible by the Taylor expansion of both sides.

Problem 2. Let $h : \mathbb{R} \to \mathbb{R}$ be a bounded, measurable function. For each n let ψ_n be a non-negative, continuous function on $\mathbb{R}$ s.t.

(i) $\psi_n(x) = 0$ for $x \leq -\frac{1}{n}$ and $x \geq 0$,

(ii) $\int_{-\infty}^{\infty} \psi_n(x) dx = 1$,

i.e. a certain Dirac-delta approximating sequence. Consider the functions

$$g_n(t) := \int_0^t \psi_n(s-t) h(s) ds. \tag{5}$$

[1] Strictly speaking for $f \in L^2(\mathbb{R})$ one should write, $\hat{f}(k) = \lim_{T\to\infty} \int_{|t|\leq T} e^{ikt} f(t) dt$, where the limit is in $L^2(\mathbb{R})$, but this is not important for solving this exercise.

Show that, for any fixed $0 \le S < T$,

$$\int_S^T (g_n(t) - h(t))^2 dt \underset{n\to\infty}{\to} 0. \tag{6}$$

Hints:

- Use support properties of ψ_n to show that for $1/n < t$

$$g_n(t) - h(t) = \int_{-\infty}^{\infty} \psi_n(s')(h(t+s') - h(t))ds'. \tag{7}$$

- Let $K := [S,T]$. Note that you can enter with the norm under the integral as follows

$$\|g_n - h\|_{L^2(K)} \le \int_{-\infty}^{\infty} \psi_n(s')\|h(\cdot + s') - h(\cdot)\|_{L^2(K)} ds' \tag{8}$$

  (Minkowski inequality) and $\|g_n - h\|^2_{L^2(K)} =$ l.h.s. of (6).

- Apply Problem 1 to show that the r.h.s. of (8) tends to zero as $n \to \infty$. Note the mismatch between bounded h and $L^2(K)$-norm in (8) and $h \in L^2(\mathbb{R})$ in Problem 1. Find a suitable reasoning to close this gap.

Solution. We write

$$\begin{aligned} g_n(t) - h(t) &= \int_0^{\infty} \psi_n(s-t)h(s)ds - h(t) \\ &= \int_{-t}^{\infty} \psi_n(s')h(t+s')ds' - \int_{-\infty}^{\infty} \psi_n(s')h(t)ds', \end{aligned} \tag{9}$$

where in the last step we used (ii). We note that for $t = 0$ we have $g_n(0) = 0$, while $h(0)$ may be different from zero. But, since $\{t = 0\}$ has Lebesgue measure zero, this is not an obstacle to L^2-convergence. Suppose that $t > 0$. Choose n so large that $1/n < t$. Then

$$\begin{aligned} (9) &= \int_{-\infty}^{\infty} \psi_n(s')h(t+s')ds' - \int_{-\infty}^{\infty} \psi_n(s')h(t)ds' \\ &= \int_{-\infty}^{\infty} \psi_n(s')(h(t+s') - h(t))ds'. \end{aligned} \tag{10}$$

Consequently, for $K := [S,T]$, by the Minkowski inequality (entering with a norm under the integral)

$$\begin{aligned} \|g_n - h\|_{L^2(K)} &\le \int_{-\infty}^{\infty} \psi_n(s')\|h(\cdot + s') - h(\,\cdot\,)\|_{L^2(K)} ds' \\ &\le \sup_{|s'|\le 1/n} \|h(\cdot + s') - h(\,\cdot\,)\|_{L^2(K)}. \end{aligned} \tag{11}$$

Let us now show that the last norm tends to zero with $s' \to 0$. Since h is bounded, $h_1(t) := \frac{1}{1+t^2} h(t)$ is in $L^2(\mathbb{R})$.

$$\begin{aligned}
\|h(\,\cdot\, + s') - h(\,\cdot\,)\|^2_{L^2(K)} &= \int dt\, \chi_K(t) |h(t+s') - h(t)|^2 \\
&= \int dt\, \chi_K(t) |(1 + (t+s')^2) h_1(t+s') - (1+t^2) h_1(t)|^2 \\
&= \int dt\, \chi_K(t) (1+t^2) |h_1(t+s') - h_1(t)|^2 \qquad (12) \\
&\quad + s' \int dt\, \chi_K(t) 2t |h_1(t+s')|^2 \qquad (13) \\
&\quad + (s')^2 \int dt\, \chi_K(t) |h_1(t+s')|^2. \qquad (14)
\end{aligned}$$

We estimate

$$(13) \le s' 2T \int dt |h_1(t)|^2, \qquad (15)$$

$$(14) \le (s')^2 \int dt\, |h_1(t)|^2. \qquad (16)$$

We have

$$(12) \le (1+T^2) \int dt\, |h_1(t+s', \omega) - h_1(t, \omega)|^2. \qquad (17)$$

This tends to zero as $s' \to 0$ by Problem 1. This concludes the proof of (6).

Problem 3. Let $\mathcal{F}$ be a finite σ-field on a set Ω. Recall that a set $A \in \mathcal{F}$ is called an atom if $A \neq \emptyset$ and for every $F \in \mathcal{F}$ such that $F \subseteq A$, one has either $F = \emptyset$ or $F = A$.

(a) Show that every $F \in \mathcal{F}$ is a union of atoms of $\mathcal{F}$.

(b) Show that every function $f : \Omega \to \mathbb{R}$, measurable w.r.t. $\mathcal{F}$, is constant on atoms.

Solution. To prove (a), define a relation on Ω by

$$x \sim y \quad \Longleftrightarrow \quad (\forall F \in \mathcal{F})\, [\, x \in F \iff y \in F\,].$$

(i) Clearly, for each $x \in \Omega$, the equivalence class of x under the above relation is

$$A(x) = \bigcap_{\substack{F \in \mathcal{F} \\ x \in F}} F.$$

Since $\mathcal{F}$ is finite, $A(x)$ is a finite intersection of sets in $\mathcal{F}$, hence $A(x) \in \mathcal{F}$. Let $\mathcal{A} = \{A(x) : x \in \Omega\}$.

(ii) Each $A \in \mathcal{A}$ is an atom. Indeed, suppose $A \in \mathcal{A}$ and $F \in \mathcal{F}$ with $\emptyset \neq F \subseteq A$. If $x, y \in A$, then $x \sim y$, so membership in F is constant on A. Thus, with $x \in A$, by moving y inside A we cannot depart from A. Since F is nonempty, it follows that $A \subseteq F$, hence $F = A$.

(iii) The atoms form a partition of Ω. By construction, $\sim$ is an equivalence relation, and the sets $A(x)$ are its equivalence classes. Therefore, the atoms are disjoint and

$$\Omega = \bigcup_{A \in \mathcal{A}} A.$$

(iv) Every $F \in \mathcal{F}$ is a union of atoms. Membership in F is constant on each atom A, so

$$F = \bigcup_{\substack{A \in \mathcal{A} \\ A \subseteq F}} A.$$

Hence every element F of the finite σ-field $\mathcal{F}$ is a disjoint union of atoms. Moreover,

$$\mathcal{F} = \left\{ \bigcup_{A \in \mathcal{A}_0} A : \mathcal{A}_0 \subseteq \mathcal{A} \right\}.$$

To prove (b), fix an atom $A \in \mathcal{F}$.

(i) Suppose, for contradiction, that f is not constant on A. Then there exist $x, y \in A$ with $f(x) \neq f(y)$.

(ii) Choose a Borel set $B \subset \mathbb{R}$ such that $f(x) \in B$ and $f(y) \notin B$; for example, take $B = (-\infty, t]$ with t strictly between $f(x)$ and $f(y)$.

(iii) Since f is $\mathcal{F}$-measurable, the set

$$F := \{\, \omega \in \Omega : \ f(\omega) \in B \,\} = f^{-1}(B)$$

belongs to $\mathcal{F}$.

(iv) We have $x \in A \cap F$ and $y \in A \setminus F$, hence $A \cap F$ is a measurable subset of A that is nonempty and not equal to A. This contradicts the assumption that A is an atom. Therefore f must be constant on A.

To be discussed in class: 28.11.2025