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Stochastic Differential Equations
Homework Sheet 7

Problem 1. Let $h \in L^2(\mathbb{R})$. Show that

$$\lim_{s\to 0} \|h(\cdot + s) - h(\,\cdot\,)\|^2_{L^2(\mathbb{R})} = \lim_{s\to 0} \int |h(t+s) - h(t)|^2 dt = 0. \tag{1}$$

Hint: One way to solve this problem is to use the Plancherel theorem: For $f \in L^2(\mathbb{R})$, the Fourier transform $\hat{f}(k) = \int \mathrm{e}^{ikt} f(t)dt$ [1] is also in $L^2(\mathbb{R})$ and satisfies

$$\|f\|_{L^2(\mathbb{R})} = \frac{1}{2\pi} \|\hat{f}\|_{L^2(\mathbb{R})}. \tag{2}$$

Problem 2. Let $h : \mathbb{R} \to \mathbb{R}$ be a bounded, measurable function. For each n let ψ_n be a non-negative, continuous function on $\mathbb{R}$ s.t.

(i) $\psi_n(x) = 0$ for $x \le -\frac{1}{n}$ and $x \ge 0$,

(ii) $\int_{-\infty}^{\infty} \psi_n(x)dx = 1$,

i.e. a certain Dirac-delta approximating sequence. Consider the functions

$$g_n(t) := \int_0^t \psi_n(s-t)h(s)ds. \tag{3}$$

Show that, for any fixed $0 \le S < T$,

$$\int_S^T (g_n(t) - h(t))^2 dt \underset{n\to\infty}{\to} 0. \tag{4}$$

Hints:

- Use support properties of ψ_n to show that for $1/n < t$
$$g_n(t) - h(t) = \int_{-\infty}^{\infty} \psi_n(s')(h(t+s') - h(t))ds'. \tag{5}$$
- Let $K := [S, T]$. Note that you can enter with the norm under the integral as follows
$$\|g_n - h\|_{L^2(K)} \le \int_{-\infty}^{\infty} \psi_n(s') \|h(\cdot + s') - h(\cdot)\|_{L^2(K)} ds' \tag{6}$$
(Minkowski inequality) and $\|g_n - h\|^2_{L^2(K)} =$ l.h.s of (4).

[1] Strictly speaking for $f \in L^2(\mathbb{R})$ one should write, $\hat{f}(k) = \lim_{T\to\infty} \int_{|t|\le T} \mathrm{e}^{ikt} f(t)dt$, where the limit is in $L^2(\mathbb{R})$, but this is not important for solving this exercise.

- Apply Problem 1 to show that the r.h.s. of (6) tends to zero as $n \to \infty$. Note the mismatch between bounded h and $L^2(K)$-norm in (6) and $h \in L^2(\mathbb{R})$ in Problem 1. Find a suitable reasoning to close this gap.

Problem 3. Let $\mathcal{F}$ be a finite σ-field on a set Ω. Recall that a set $A \in \mathcal{F}$ is called an atom if $A \neq \emptyset$ and for every $F \in \mathcal{F}$ such that $F \subseteq A$, one has either $F = \emptyset$ or $F = A$.

(a) Show that every $F \in \mathcal{F}$ is a union of atoms of $\mathcal{F}$.

(b) Show that every function $f : \Omega \to \mathbb{R}$, measurable w.r.t. $\mathcal{F}$, is constant on atoms.

To be discussed in class: 28.11.2025