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Stochastic Differential Equations
Homework Sheet 8 - solutions

In this Homework Sheet $\{B_t\}_{t\in\mathbb{R}_+}$ *denotes one-dimensional Brownian motion with* $B_0 = 0$ *and continuous paths. We set, like in class,* $B_j := B_{t_j}$, $\Delta B_j := B_{j+1} - B_j$, *where*

$$t_j := t_j^{(n)} := \begin{cases} j2^{-n} & \text{if} \quad S \leq j2^{-n} \leq T, \\ S & \text{if} \quad j2^{-n} < S, \\ T & \text{if} \quad j2^{-n} > T. \end{cases} \tag{1}$$

The overall goal of this Homework Sheet is to compute $\int_0^T B_t^2 dB_t$ *from the definition of the Itô integral.*

Problem 1. Prove from the definition of the Itô integral that

$$\int_0^T t dB_t = TB_T - \int_0^T B_t dt. \tag{2}$$

Hint 1. Check that $\sum_j \Delta(t_j B_j) = \sum_j t_j \Delta B_j + \sum_j B_{j+1}\Delta t_j$.

Hint 2. It may be helpful to remember that L^2-convergence and almost sure convergence are unrelated. However, each of them separately implies convergence in probability.

Solution. First, we note that

$$\begin{aligned}\sum_j (t_j \Delta B_j + B_{j+1}\Delta t_j) &= \sum_j \big(t_j(B_{j+1} - B_j) + B_{j+1}(t_{j+1} - t_j)\big) \\ &= \sum_j (t_{j+1}B_{j+1} - t_j B_j) = \sum_j \Delta(t_j B_j). \end{aligned} \tag{3}$$

Now put $\varphi_n(t,\omega) = \sum_j t_j \cdot \chi_{[t_j,t_{j+1})}(t)$ (with trivial dependence on ω). We compute

$$\begin{aligned} E\Big[\int_0^T (\varphi_n - t)^2 dt\Big] &= \int_0^T (\sum_j t_j \cdot \chi_{[t_j,t_{j+1})}(t) - t)^2 dt \\ &= \sum_j \int_{t_j}^{t_{j+1}} (t_j - t)^2 dt \\ &= \sum_j \int_0^{t_{j+1}-t_j} \tau^2 d\tau = \frac{1}{3}\sum_j (t_{j+1} - t_j)^3 \underset{n\to\infty}{\to} 0, \end{aligned} \tag{4}$$

where in the last step we restricted attention to $j \leq 2^n T$, and used $(t_{j+1} - t_j)^3 = 2^{-3n}$. Consequently, by (3),

$$\begin{aligned}\int_0^T t dB_t =& \lim_{n\to\infty} \sum_j t_j \Delta B_j \\ =& \lim_{n\to\infty} \sum_j \Delta(t_j B_j) - \lim_{n\to\infty} \sum_j B_{j+1} \Delta t_j \\ =& T B_T - \int_0^T B_t dt. \end{aligned} \tag{5}$$

Here in the last step we used that:

- $\lim_{n\to\infty} \sum_j B_{j+1}\Delta t_j$ exists in L^2, because $\lim_{n\to\infty} \sum_j t_j \Delta B_j$ exists in L^2 by the theory of Itô integral.
- $\lim_{n\to\infty} \sum_j B_{j+1}\Delta t_j$ exists pointwise in ω as a Riemann integral and equals $\int_0^T B_t dt$.

In general, there is no implication between L^2 and pointwise convergence. But each of these two types of convergence separately implies convergence in probability (to the same limit) cf. HS3, HS9. As the latter topology is given by a metric, the limit is unique and must be equal to $\int_0^T B_t dt$.

Problem 2. Show that

$$\lim_{n\to\infty} \sum_j B_j \Delta t_j \underset{n\to\infty}{\to} \int_0^T B_t dt \tag{6}$$

in $L^2(\Omega, P)$. Hint: From your solution to Problem 1 you may see that $\lim_{n\to\infty} \sum_j B_{j+1}\Delta t_j = \int_0^T B_t dt$ in $L^2(\Omega, P)$.

Solution. By the hint, it suffices to estimate

$$\begin{aligned} E\big[\big(\sum_j \Delta B_j \Delta t_j\big)^2\big] =& \sum_{j,k} \Delta t_j \Delta t_k E\big[\Delta B_j \Delta B_k\big] \\ =& \sum_j (\Delta t_j)^2 E\big[(\Delta B_j)^2\big] = \sum_j (\Delta t_j)^3 = \sum_{j \leq T2^n} 2^{-3n} \underset{n\to\infty}{\to} 0. \end{aligned} \tag{7}$$

where we used the independence of increments.

Problem 3. Prove that

$$\lim_{n\to\infty} \sum_j B_j (\Delta B_j)^2 \underset{n\to\infty}{\to} \int_0^T B_t dt. \tag{8}$$

in $L^2(\Omega, P)$. Hint 1: Decompose

$$\sum_j B_j (\Delta B_j)^2 = \sum_j B_j [(\Delta B_j)^2 - \Delta t_j] + \sum_j B_j \Delta t_j. \tag{9}$$

Show that the first sum on the r.h.s. of (9) tends to zero in $L^2(\Omega, P)$. Then apply Problem 2 to the second sum.

Hint 2: Recall that by HS3, Problem 1, $E(B_t^4) = 3t^2$.

Solution. Define $D := \sum_j B_j((\Delta B_j)^2 - \Delta t_j) =: \sum_j D_j$ and compute

$$E[D^2] = \sum_{j,k} E[D_j D_k] = \sum_j E[D_j^2] + \sum_{j \neq k} E[D_j D_k]. \tag{10}$$

(i) Let us first analyze $E[D_j D_k]$ for $j \neq k$. Suppose, for concreteness, that $j < k$. Then $(\Delta B_k)^2 - \Delta t_k$ is independent of $\sigma(B_s : s \leq t_k)$. Consequently, it is independent not only of $B_j, B_k, ((\Delta B_j)^2 - \Delta t_j)$ separately, but also of their product $B_j B_k ((\Delta B_j)^2 - \Delta t_j)$. Hence,

$$E[D_j D_k] = E[(\Delta B_k)^2 - \Delta t_k] E[B_j B_k ((\Delta B_j)^2 - \Delta t_j)] = 0, \tag{11}$$

since $E[(\Delta B_k)^2] = \Delta t_k$. Thus all the off-diagonal terms vanish.

(ii) Let us now show that

$$E[D_j^2] = 2t_j (\Delta t_j)^2. \tag{12}$$

Using independence of B_j and ΔB_j, we write

$$E[|D_j|^2] = E\big[B_j^2 [(\Delta B_j)^2 - \Delta t_j]^2\big] = E[B_j^2]\, E[[(\Delta B_j)^2 - \Delta t_j]^2]. \tag{13}$$

We have $E[B_j^2] = t_j$. Using $E[(\Delta B_j)^2] = \Delta t_j$, $E[(\Delta B_j)^4] = 3(\Delta t_j)^2$, we obtain

$$\begin{aligned} E[((\Delta B_j)^2 - \Delta t_j)^2] &= E[(\Delta B_j)^4] - 2\Delta t_j E[(\Delta B_j)^2] + (\Delta t_j)^2 \\ &= 3(\Delta t_j)^2 - 2(\Delta t_j)^2 + (\Delta t_j)^2 = 2(\Delta t_j)^2, \end{aligned} \tag{14}$$

which gives (12).

(iii) Summing up, we have

$$E[D^2] = 2 \sum_{j \leq T2^n} t_j (\Delta t_j)^2 = 2 \sum_{j \leq T2^n} j 2^{-n} 2^{-2n} \underset{n \to \infty}{\to} 0. \tag{15}$$

Problem 4. Prove that

$$\sum_j (\Delta B_j)^3 \underset{n \to \infty}{\to} 0 \tag{16}$$

in $L^2(\Omega, P)$. Hint: $E[(\Delta B_j)^6] = 15(\Delta t_j)^3$.

Solution. We estimate, using independence of increments and $E[(\Delta B_j)^3] = 0$,

$$\begin{aligned} E[(\sum_j (\Delta B_j)^3)^2] &= \sum_{j,k} E[(\Delta B_j)^3 (\Delta B_k)^3] = \sum_j E[(\Delta B_j)^6] \\ &= 15 \sum_j (\Delta t_j)^3 = 15 \sum_{j \leq 2^n T} 2^{-3n} \underset{n \to \infty}{\to} 0, \end{aligned} \tag{17}$$

which concludes the proof.

Problem 5. Prove from the definition of the Itô integral that

$$\int_0^T B_t^2 dB_t = \frac{1}{3} B_T^3 - \int_0^T B_t dt. \tag{18}$$

Hint: Follow the overall strategy from the computation of $\int_0^T B_t dB_t$ in class and use the information from the problems above as required.

Solution. First, suppose that $0 \le s \le t$ and compute

$$\begin{aligned} E[B_s^2 B_t^2] &= E[B_s^2((B_t - B_s) + B_s)^2] \\ &= E[B_s^2((B_t - B_s)^2 + 2(B_t - B_s)B_s + B_s^2)] \\ &= E[B_s^2]E[(B_t - B_s)^2] + 2E[B_t - B_s]E[B_s] + E[B_s^4] \\ &= s(t-s) + 3s^2 = st + 2s^2, \end{aligned} \tag{19}$$

where we used that independent random variables are uncorrelated and $E(B_s^4) = 3s^2$. Consequently,

$$\begin{aligned} E[(B_s^2 - B_t^2)^2] &= E[(B_s^4 - 2B_s^2 B_t^2 + B_t^4)] = -2(st + 2s^2) + 3s^2 + 3t^2 \\ &= -s^2 - 2st + 3t^2 = 3(t-s)^2 + 4s(t-s). \end{aligned} \tag{20}$$

After these preparations, we put $\varphi_n(t,\omega) = \sum_j B_j^2(\omega) \cdot \chi_{[t_j, t_{j+1})}(t)$. Then, by (20),

$$\begin{aligned} E\left[\int_0^T (\varphi_n - B_t^2)^2 dt\right] &= E\left[\sum_j \int_{t_j}^{t_{j+1}} (B_j^2 - B_t^2)^2 dt\right] \\ &= \sum_j \int_{t_j}^{t_{j+1}} \{3(t - t_j)^2 + 4t_j(t - t_j)\} dt \\ &= \sum_j (t_{j+1} - t_j)^3 + 2\sum_j t_j (t_{j+1} - t_j)^2. \end{aligned} \tag{21}$$

Recall that $t_j = j2^{-n} \le T$. Then

$$\begin{aligned} (21) &\le \sum_{j2^{-n} \le T} [(t_{j+1} - t_j)^3 + 2t_j(t_{j+1} - t_j)^2] \\ &= \sum_{j \le T2^n} [2^{-3n} + 2j2^{-n}2^{-2n}] \underset{n\to\infty}{\to} 0. \end{aligned} \tag{22}$$

Consequently,

$$\int_0^T B_t^2 dB_t = \lim_{n\to\infty} \int_0^T \varphi_n dB_t = \lim_{n\to\infty} \sum_j B_j^2 \Delta B_j. \tag{23}$$

Note that

$$\Delta(B_j^3) = 3B_j^2 \Delta B_j + 3B_j(\Delta B_j)^2 + (\Delta B_j)^3. \tag{24}$$

By applying $\sum_j$ to both sides and using Problems 3,4 we conclude the proof.