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Stochastic Differential Equations
Homework Sheet 8

In this Homework Sheet $\{B_t\}_{t\in\mathbb{R}_+}$ *denotes one-dimensional Brownian motion with* $B_0 = 0$ *and continuous paths. We set, like in class,* $B_j := B_{t_j}$, $\Delta B_j := B_{j+1} - B_j$, *where*

$$t_j := t_j^{(n)} := \begin{cases} j2^{-n} & \text{if} \quad S \le j2^{-n} \le T, \\ S & \text{if} \quad j2^{-n} < S, \\ T & \text{if} \quad j2^{-n} > T. \end{cases} \tag{1}$$

The overall goal of this Homework Sheet is to compute $\int_0^T B_t^2 dB_t$ *from the definition of the Itô integral.*

Problem 1. Prove from the definition of the Itô integral that

$$\int_0^T t dB_t = TB_T - \int_0^T B_t dt. \tag{2}$$

Hint 1. Check that $\sum_j \Delta(t_j B_j) = \sum_j t_j \Delta B_j + \sum_j B_{j+1}\Delta t_j$.

Hint 2. It may be helpful to remember that L^2-convergence and almost sure convergence are unrelated. However, each of them separately implies convergence in probability.

Problem 2. Show that

$$\lim_{n\to\infty} \sum_j B_j \Delta t_j \underset{n\to\infty}{\to} \int_0^T B_t dt \tag{3}$$

in $L^2(\Omega, P)$. Hint: From your solution to Problem 1 you may see that $\lim_{n\to\infty} \sum_j B_{j+1}\Delta t_j = \int_0^T B_t dt$ in $L^2(\Omega, P)$.

Problem 3. Prove that

$$\lim_{n\to\infty} \sum_j B_j (\Delta B_j)^2 \underset{n\to\infty}{\to} \int_0^T B_t dt. \tag{4}$$

in $L^2(\Omega, P)$. Hint 1: Decompose

$$\sum_j B_j (\Delta B_j)^2 = \sum_j B_j[(\Delta B_j)^2 - \Delta t_j] + \sum_j B_j \Delta t_j. \tag{5}$$

Show that the first sum on the r.h.s. of (5) tends to zero in $L^2(\Omega, P)$. Then apply Problem 2 to the second sum.

Hint 2: Recall that by HS3, Problem 1, $E(B_t^4) = 3t^2$.

Problem 4. Prove that

$$\sum_j (\Delta B_j)^3 \underset{n\to\infty}{\to} 0 \tag{6}$$

in $L^2(\Omega, P)$. Hint: $E[(\Delta B_j)^6] = 15(\Delta t_j)^3$.

Problem 5. Prove from the definition of the Itô integral that

$$\int_0^T B_t^2 dB_t = \frac{1}{3} B_T^3 - \int_0^T B_t dt. \tag{7}$$

Hint: Follow the overall strategy from the computation of $\int_0^T B_t dB_t$ in class and use the information from the problems above as required.

To be discussed in class: 05.12.2025