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Stochastic Differential Equations
Homework Sheet 9 - solutions

Problem 1. Let $(\Omega, \mathcal{F}, P)$ and be a probability space and $X : \Omega \to \mathbb{R}$ a random variable. Denote by $\mu_X(F) = P(X \in F)$, $F \in \mathcal{B}(\mathbb{R})$, the law of X. Suppose that μ_X is absolutely continuous w.r.t. some σ-finite reference measure ν. Define the density of X w.r.t. ν by the Radon-Nikodym derivative

$$p_X(x) = \frac{d\mu_X}{d\nu}(x). \tag{1}$$

This is consistent with the usual definition of the density, since

$$\mu_X(F) = \int_F \frac{d\mu_X}{d\nu}(x) d\nu(x). \tag{2}$$

Prove the following facts:

(a) Let ν be the counting measure, i.e., $\nu(F) = \#\{x \in \mathbb{Z} : x \in F\}$. Then,

$$p_X(x) = \mu_X(\{x\}) = P(X = x) = E[\chi_{\{X=x\}}]. \tag{3}$$

(b) Let ν be the Lebesgue measure. Then,

$$p_X(x) = \lim_{\varepsilon\to 0} \frac{\mu_X([x-\varepsilon, x+\varepsilon])}{2\varepsilon} = \lim_{\varepsilon\to 0} \frac{P(X \in [x-\varepsilon, x+\varepsilon])}{2\varepsilon} = \lim_{\varepsilon\to 0} E\big[\frac{1}{2\varepsilon}\chi_{[x-\varepsilon,x+\varepsilon]}(X)\big] \tag{4}$$

You can add some regularity assumptions on p_X, if it helps (continuity, differentiability, etc.)

(Side remark: Note that $\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\chi_{[x-\varepsilon,x+\varepsilon]}(y) = \delta(y-x)$ in $D'(\mathbb{R})$, so in a sloppy notation $p_X(x) = E[\delta(X - x)]$, which allows for a comparison with (3).)

Solution. We check (a) by substituting $p_X(x) = \frac{d\mu_X}{d\nu}(x)$ to formula (2).

$$\int_F \frac{d\mu_X}{d\nu}(x) d\nu(x) = \int_F \mu_X(\{x\}) d\nu(x) = \sum_{x\in\mathbb{Z}, x\in F} \mu_X(\{x\}) = \mu_X(F). \tag{5}$$

Part (b) in fact amounts to the 'Lebesgue differentiation theorem'. But, assuming continuity of p_X, we can give an elementary proof. By (2):

$$\frac{\mu_X([x-\varepsilon, x+\varepsilon])}{2\varepsilon} = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} p_X(x+\tilde{x}) d\tilde{x} = p_X(x+t_\varepsilon), \tag{6}$$

where we applied the mean value theorem and introduced $t_\varepsilon \in [-\varepsilon, \varepsilon]$. By taking $\lim_{\varepsilon\to 0}$ of both sides, we complete the proof.

Problem 2. Consider the probability space $(\Omega, \mathcal{F}, P) = ([0, L], \mathcal{B}([0, L]), \frac{1}{L}dx)$, $L \in \mathbb{N}$, $L \geq 2$. Let $Z(x) := \sin(2\pi x)$, $X_a(x) := \chi_{[a,a+1]}(x)$ and $Y_a(x) = \chi_{[a,a+1/2]}(x)$ for $a \in [0, L-1]$.

(a) Show that Z is independent of X_a for any $a \in [0, L-1]$.

(b) Show that Z is not independent of $Y_{1/2}$.

Since $Y_{1/2} = X_0 X_{1/2}$ this shows that independence is not preserved under taking products.

Solution. Regarding (a), it suffices to show that $\sigma(Z)$ is independent of $\sigma(X_a)$. Let $F \subset \mathbb{R}$ be an interval and note that $\sigma(X_a) = \{\emptyset, [a, a+1], [a, a+1]^c, \Omega\}$. We compute

$$\begin{aligned} P(Z^{-1}(F) \cap [a, a+1]) &= \int_0^L \chi_F(Z(x))\chi_{[a,a+1]}(x)\frac{dx}{L} \\ &= \int_a^{a+1} \chi_F(Z(x))\frac{dx}{L} = \int_0^1 \chi_F(Z(x))\frac{dx}{L}. \end{aligned} \tag{7}$$

Here in the last step we used that Z has period 1, in which case it does not matter over which interval of length 1 we integrate it. It follows from the latter statement, that averaging over multiple periods amounts to averaging over one period, so

$$P(Z^{-1}(F)) = \frac{1}{L}\int_0^L \chi_F(Z(x))dx = \int_0^1 \chi_F(Z(x))dx. \tag{8}$$

Noting that

$$P([a, a+1]) = \frac{1}{L}, \tag{9}$$

we get from the above three equations that $P(Z^{-1}(F) \cap [a, a+1]) = P(Z^{-1}(F))P([a, a+1])$. $P(Z^{-1}(F) \cap [a, a+1]^c) = P(Z^{-1}(F))P([a, a+1]^c)$ is automatic. (In fact, if A is independent of B then A is independent of B^c).

Regarding (b), it suffices to show that Z and $Y_{1/2}$ are correlated. We have

$$E[Z] = 0, \quad E[Y_{1/2}] = \frac{1}{L}\int_{1/2}^1 dx = \frac{1}{2L}, \quad E[ZY_{1/2}] = \frac{1}{L}\int_{1/2}^1 \sin(2\pi x)dx = -\frac{1}{L\pi}. \tag{10}$$

Thus $E[ZY_{1/2}] \neq E[Z]E[Y_{1/2}]$.

Problem 3. Let $\{X_n\}_{n\geq 1}$ and X be random variables on a common probability space. We say that $\{X_n\}_{n\in\mathbb{N}}$ *converges in probability* to X, written $X_n \xrightarrow{P} X$, if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0.$$

Show that if $X_n \to X$ almost surely, i.e.,

$$X_n(\omega) \to X(\omega), \quad \text{for} \quad \omega \in \Omega \backslash N, \quad \text{for some } N \in \mathcal{F} \text{ s.t. } P(N) = 0,$$

then $X_n \xrightarrow{P} X$.

Solution. Fix $\varepsilon > 0$. Then:

$$P(|X_n - X| > \varepsilon) = \int \chi_{[\varepsilon,\infty)}(|X_n(\omega) - X(\omega)|) dP(\omega). \tag{11}$$

Now the statement follows from almost sure convergence by the dominated convergence and the fact that $\chi_{[\varepsilon,\infty)}(|X_n(\omega) - X(\omega)|) = 0$ for sufficiently large n.

Side remark: Almost sure convergence does not come from a topology. This follows from the following facts:

(i) Fact: There exists a sequence $\{Y_n\}_{n\in\mathbb{N}}$ which converges to zero in probability but does not converge almost surely.

(ii) Then there is a subsequence $\{Y_{n(k)}\}_{k\in\mathbb{N}}$ which is outside some $N(0)$, where $N(0)$ denote neighbourhoods of 0 in the hypothetical topology of a.s. convergence.

(iii) Theorem: If $\{X_n\}_{n\in\mathbb{N}}$ converges in probability it has a subsequence $\{X_{n(k)}\}_{k\in\mathbb{N}}$ which converges almost surely.

(iv) Thus there is a subsequence $\{Y_{n(k(\ell))}\}_{\ell\in\mathbb{N}}$ converging almost surely to zero, which is a contradiction.

Problem 4. Let $(\Omega, \mathcal{F}, P) = ([-1,1], \mathcal{B}([-1,1]), dx/2)$, where dx denotes Lebesgue measure on $[-1,1]$. Let $Y(x) = |x|$. Compute $E[X|\sigma(Y)]$ for L^2 random variables X. Determine $E[X|\sigma(Y)]$ for $X(x) = \mathrm{e}^x$ as an explicit function of x.

Solution. We proceed in steps:

(i) We first describe $\sigma(Y)$. By the Doob–Dynkin lemma, a random variable Z is $\sigma(Y)$-measurable if and only if there exists a measurable function $g : [0,1] \to \mathbb{R}$ such that

$$Z(x) = g(Y(x)) = g(|x|) \quad \text{for almost all } x \in [-1,1].$$

For such Z we have

$$Z(-x) = g(|-x|) = g(|x|) = Z(x),$$

so Z is an even function.

Conversely, if $f \in L^2([-1,1], dx/2)$ is even, then $f(x) = f(|x|)$ Therefore,

$$L^2(\sigma(Y)) = \{f \in L^2([-1,1]) : f(-x) = f(x) \text{ a.e.}\}.$$

Thus $L^2(\sigma(Y))$ is the closed subspace of even functions.

(ii) We compute $E[X \mid \sigma(Y)]$.

In L^2, the conditional expectation $E[X \mid \sigma(Y)]$ is the orthogonal projection of X onto the closed subspace $L^2(\sigma(Y))$ of even functions.

For any $X \in L^2([-1,1], dx/2)$, define

$$X_e(x) := \frac{X(x) + X(-x)}{2}, \qquad X_o(x) := \frac{X(x) - X(-x)}{2}.$$

Then $X = X_e + X_o$, where X_e is even and X_o is odd.

(iii) We show that X_e is the orthogonal projection of X onto the even subspace.

Let f be any even function in $L^2([-1,1], dx/2)$. Consider the inner product

$$\langle X_o, f \rangle = \int_{-1}^{1} X_o(x) f(x) \, \frac{dx}{2}.$$

Since X_o is odd and f is even, their product $X_o(x)f(x)$ is odd. As the measure $dx/2$ is symmetric, we obtain

$$\int_{-1}^{1} X_o(x) f(x) \, \frac{dx}{2} = 0.$$

Thus X_o is orthogonal to every even function, and hence X_e is the orthogonal projection of X onto $L^2(\sigma(Y))$.

(iv) Therefore,

$$E[X \mid \sigma(Y)] = X_e.$$

Explicitly,

$$E[X \mid \sigma(Y)](x) = \frac{X(x) + X(-x)}{2} \quad \text{for almost all } x \in [-1,1].$$

In particular, for $X(x) = \mathrm{e}^x$, we have $E[X \mid \sigma(Y)](x) = \cosh(x)$.

To be discussed in class: 12.12.2025