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Stochastic Differential Equations
Homework Sheet 9

Problem 1. Let $(\Omega, \mathcal{F}, P)$ and be a probability space and $X : \Omega \to \mathbb{R}$ a random variable. Denote by $\mu_X(F) = P(X \in F)$, $F \in \mathcal{B}(\mathbb{R})$, the law of X. Suppose that μ_X is absolutely continuous w.r.t. some σ-finite reference measure ν. Define the density of X w.r.t. ν by the Radon-Nikodym derivative

$$p_X(x) = \frac{d\mu_X}{d\nu}(x). \tag{1}$$

This is consistent with the usual definition of the density, since

$$\mu_X(F) = \int_F \frac{d\mu_X}{d\nu}(x) d\nu(x). \tag{2}$$

Prove the following facts:

(a) Let ν be the counting measure, i.e., $\nu(F) = \#\{x \in \mathbb{Z} : x \in F\}$. Then,

$$p_X(x) = \mu_X(\{x\}) = P(X = x) = E[\chi_{\{X=x\}}]. \tag{3}$$

(b) Let ν be the Lebesgue measure. Then,

$$p_X(x) = \lim_{\varepsilon \to 0} \frac{\mu_X([x-\varepsilon, x+\varepsilon])}{2\varepsilon} = \lim_{\varepsilon \to 0} \frac{P(X \in [x-\varepsilon, x+\varepsilon])}{2\varepsilon} = \lim_{\varepsilon \to 0} E\Big[\frac{1}{2\varepsilon}\chi_{[x-\varepsilon,x+\varepsilon]}(X)\Big] \tag{4}$$

You can add some regularity assumptions on p_X, if it helps (continuity, differentiability, etc.)

(Side remark: Note that $\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon}\chi_{[x-\varepsilon,x+\varepsilon]}(y) = \delta(y-x)$ in $D'(\mathbb{R})$, so in a sloppy notation $p_X(x) = E[\delta(X-x)]$, which allows for a comparison with (3).)

Problem 2. Consider the probability space $(\Omega, \mathcal{F}, P) = ([0, L], \mathcal{B}([0, L]), \frac{1}{L}dx)$, $L \in \mathbb{N}$, $L \geq 2$. Let $Z(x) := \sin(2\pi x)$, $X_a(x) := \chi_{[a,a+1]}(x)$ and $Y_a(x) = \chi_{[a,a+1/2]}(x)$ for $a \in [0, L-1]$.

(a) Show that Z is independent of X_a for any $a \in [0, L-1]$.

(b) Show that Z is not independent of $Y_{1/2}$.

Since $Y_{1/2} = X_0 X_{1/2}$ this shows that independence is not preserved under taking products.

Problem 3. Let $\{X_n\}_{n\geq 1}$ and X be random variables on a common probability space. We say that $\{X_n\}_{n\in\mathbb{N}}$ *converges in probability* to X, written $X_n \xrightarrow{P} X$, if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0.$$

Show that if $X_n \to X$ almost surely, i.e.,

$$X_n(\omega) \to X(\omega), \quad \text{for} \quad \omega \in \Omega \backslash N, \quad \text{for some } N \in \mathcal{F} \text{ s.t. } P(N) = 0,$$

then $X_n \xrightarrow{P} X$.

Problem 4. Let $(\Omega, \mathcal{F}, P) = ([-1,1], \mathcal{B}([-1,1]), dx/2)$, where dx denotes Lebesgue measure on $[-1,1]$. Let $Y(x) = |x|$. Compute $E[X|\sigma(Y)]$ for L^2 random variables X. Determine $E[X|\sigma(Y)]$ for $X(x) = \mathrm{e}^x$ as an explicit function of x.

To be discussed in class: 19.12.2025