

Lectures on mathematical foundations of QFT

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Introduction

The central question of these mathematical lectures is the following:

- Is QFT logically consistent?

Although it may not seem so, this question is quite relevant for physics. For example, if QFT contained a contradiction and, say, the magnetic moment of the electron could be computed in two different ways giving two completely different results, which of them should be compared with experiments? It turns out that such a situation is not completely ruled out in QFT, since we don't have enough control over the convergence of the perturbative series. If we take first few terms of this series, we often obtain excellent agreement with experiments. But if we managed to compute all of them and sum them up, most likely the result would be infinity.

For this and other reasons, the problem of logical consistency of QFT fascinated generations of mathematical physicists. They managed to solve it only in toy models, but built impressive mathematical structures some of which I will try to explain in these lectures.

The strategy to study the logical consistency of QFT can be summarized as follows: Take QFT as presented in the physics part of this course. Take the whole mathematics with its various sub-disciplines. Now try to 'embed' QFT into mathematics, where the problem of logical consistency is under good control. The 'image' of this embedding will be some subset of mathematics which can be called Mathematical QFT. It intersects with many different sub-disciplines including algebra, analysis, group theory, measure theory and many others. It differs from the original QFT in several respects: First, some familiar concepts from the physical theory will not reappear on the mathematics side, as tractable mathematical counterparts are missing. Second, many concepts from mathematics will enter the game, some of them without direct physical meaning (e.g. different notions of continuity and convergence). Their role is to control logical consistency within mathematics.

It should also be said that the efforts to 'embed' QFT into mathematics triggered a lot of new mathematical developments. Thus advancing mathematics is another important source of motivation to study Mathematical QFT.

1 Wightman quantum field theory

The main references for this section are [1, Section VIII], [2, Section IX.8,X.7].

1.1 Relativistic Quantum Mechanics

We consider a quantum theory given by a Hilbert space \mathcal{H} (a space with a scalar product $\langle \cdot | \cdot \rangle$, which is complete in the norm $\| \cdot \| = \sqrt{\langle \cdot | \cdot \rangle}$) and:

- (a) Observables $\{O_i\}_{i \in \mathbb{I}}$. Hermitian / self-adjoint operators.
- (b) Symmetry transformations $\{U_j\}_{j \in \mathbb{J}}$. Unitary/anti-unitary operators.

1.1.1 Observables

1. Consider an operator $O : D(O) \rightarrow \mathcal{H}$ i.e. a linear map from a dense domain $D(O) \subset \mathcal{H}$ to \mathcal{H} . $D(O) = \mathcal{H}$ only possible for bounded operators O (i.e. with bounded spectrum). In other words, $O \in \mathcal{L}(D(O), \mathcal{H})$, which is the space of linear maps between the two spaces.
2. $D(O^\dagger) := \{ \Psi \in \mathcal{H} \mid |\langle \Psi | O \Psi' \rangle| \leq c_\Psi \| \Psi' \| \text{ for all } \Psi' \in D(O) \}$. Thereby, $O^\dagger \Psi$ is well defined for any $\Psi \in D(O^\dagger)$ via the Riesz theorem.
3. We say that O is Hermitian, if $D(O) \subset D(O^\dagger)$ and $O^\dagger \Psi = O \Psi$ for all $\Psi \in D(O)$.
4. We say that O is self-adjoint, if it is Hermitian and $D(O) = D(O^\dagger)$. Advantage: we can define e^{itO} and then also a large class of other functions via the Fourier transform. E.g. $f(O) = (2\pi)^{-1/2} \int dt e^{itO} \check{f}(t)$ for $f \in C_0^\infty(\mathbb{R})$ (smooth, compactly supported, complex-valued).
5. We say that operators O_1, O_2 weakly commute on some dense domain $D \subset D(O_1) \cap D(O_2) \cap D(O_1^\dagger) \cap D(O_2^\dagger)$ if for all $\Psi_1, \Psi_2 \in D$

$$\begin{aligned} 0 &= \langle \Psi_1 | [O_1, O_2] \Psi_2 \rangle = \langle \Psi_1 | O_1 O_2 \Psi_2 \rangle - \langle \Psi_1 | O_2 O_1 \Psi_2 \rangle \\ &= \langle O_1^\dagger \Psi_1 | O_2 \Psi_2 \rangle - \langle O_2^\dagger \Psi_1 | O_1 \Psi_2 \rangle. \end{aligned} \quad (1)$$

6. We say that two self-adjoint operators O_1, O_2 strongly commute, if

$$[e^{it_1 O_1}, e^{it_2 O_2}] = 0 \text{ for all } t_1, t_2 \in \mathbb{R}. \quad (2)$$

No domain problems here, since e^{itO} is always bounded, hence $D(e^{itO}) = \mathcal{H}$.

7. Let O_1, \dots, O_n be a family of self-adjoint operators which mutually strongly commute. For any $f \in C_0^\infty(\mathbb{R}^n)$ we define

$$f(O_1, \dots, O_n) = (2\pi)^{-n/2} \int dt_1 \dots dt_n e^{it_1 O_1} \dots e^{it_n O_n} \check{f}(t_1, \dots, t_n). \quad (3)$$

Definition 1.1 [4] *The joint spectrum $\text{Sp}(O_1, \dots, O_n)$ of such a family of operators is defined as follows: $p \in \text{Sp}(O_1, \dots, O_n)$ if for any open neighbourhood V_p of this point there is a function $f \in C_0^\infty(\mathbb{R}^n)$ s.t. $\text{supp} f \subset V_p$ and $f(O_1, \dots, O_n) \neq 0$.*

It is easy to see that for one operator O with purely point spectrum (e.g. the Hamiltonian of the harmonic oscillator) $\text{Sp}(O)$ is the set of all the eigenvalues. But the above definition captures also the continuous spectrum without using ‘generalized eigenvectors’.

1.1.2 Symmetry transformations

We treat today only symmetries implemented by unitaries.

1. A linear bijection $U : \mathcal{H} \rightarrow \mathcal{H}$ is called a unitary if $\langle U\Psi_1 | U\Psi_2 \rangle = \langle \Psi_1 | \Psi_2 \rangle$ for all $\Psi_1, \Psi_2 \in \mathcal{H}$. We denote by $\mathcal{U}(\mathcal{H})$ the group of all unitaries on \mathcal{H} . Unitaries are suitable to describe symmetries as they preserve transition amplitudes of physical processes.
2. The Minkowski spacetime is invariant under Poincaré transformations $x \mapsto \Lambda x + a$, where $(\Lambda, a) \in \mathcal{P}_+^\uparrow$ (the proper orthochronous Poincaré group). We consider a unitary representation of this group on \mathcal{H} , i.e. a map $\mathcal{P}_+^\uparrow \ni (\Lambda, a) \mapsto U(\Lambda, a) \in \mathcal{U}(\mathcal{H})$ with the property

$$U(\Lambda_1, a_1)U(\Lambda_2, a_2) = U((\Lambda_1, a_1)(\Lambda_2, a_2)), \quad (4)$$

i.e. a group homomorphism.

3. We say that such a representation is continuous, if $(\Lambda, a) \mapsto \langle \Psi_1 | U(\Lambda, a)\Psi_2 \rangle \in \mathbb{C}$ is a continuous function for any $\Psi_1, \Psi_2 \in \mathcal{H}$.

1.1.3 Energy-momentum operators and the spectrum condition

The following fact is an immediate consequence of the Stone theorem and continuity is crucial here:

Theorem 1.2 *Given a continuous unitary representation of translations $\mathbb{R}^4 \ni a \mapsto U(a) := U(I, a) \in \mathcal{U}(\mathcal{H})$, there exist four strongly commuting self-adjoint operators P_μ , $\mu = 0, 1, 2, 3$, s.t.*

$$U(a) = e^{iP_\mu a^\mu}. \quad (5)$$

We call $P = \{P_0, P_1, P_2, P_3\}$ the energy-momentum operators.

In physical theories P_μ are unbounded operators (since values of the energy-momentum can be arbitrarily large), defined on some domains $D(P_\mu) \subset \mathcal{H}$. However, to guarantee stability of physical systems, the energy should be bounded from below in any inertial system. The mathematical formulation is the spectrum condition:

Definition 1.3 We say that a Poincaré covariant quantum theory satisfies the spectrum condition if

$$\text{Sp } P := \text{Sp}(P_0, P_1, P_2, P_3) \subset \bar{V}_+, \quad (6)$$

where $\bar{V}_+ := \{ (p_0, \vec{p}) \in \mathbb{R}^4 \mid p_0 \geq |\vec{p}| \}$ is the closed future lightcone.

1.1.4 Vacuum state

1. A unit vector $\Omega \in \mathcal{H}$ is called the vacuum state if $U(\Lambda, a)\Omega = \Omega$ for all $(\Lambda, a) \in \mathbb{R}^4$. This implies $P_\mu\Omega = 0$ for $\mu = 0, 1, 2, 3$.
2. By the spectrum condition, Ω is the ground state of the theory.
3. We say that the vacuum is unique, if Ω is the only such vector in \mathcal{H} up to multiplication by a phase.

1.1.5 Relativistic Quantum Mechanics: Summary

Definition 1.4 A relativistic quantum mechanics is given by

1. Hilbert space \mathcal{H} .
2. A continuous unitary representation $\mathcal{P}_+^\uparrow \ni (\Lambda, a) \mapsto U(\Lambda, a) \in \mathcal{U}(\mathcal{H})$ satisfying the spectrum condition.
3. Observables $\{O_i\}_{i \in \mathbb{I}}$, including P_μ .

Furthermore, \mathcal{H} may contain a vacuum vector Ω (unique or not).

So far in our collection of observables $\{O_i\}_{i \in \mathbb{I}}$ we have identified only global quantities like P_μ . (For example, to measure P_0 , we would have to add up the energies of all the particles in the universe of our theory). But actual measurements are performed locally, i.e. in bounded regions of spacetime and we would like to include the corresponding observables. We have to do it in a way which is consistent with Poincaré symmetry, spectrum condition and locality (Einstein causality). This is the role of quantum fields.

1.2 Quantum fields as operator-valued distributions

1.2.1 Tempered distributions

We recall some definitions:

1. The Schwartz-class functions:

$$S = \{ f \in C^\infty(\mathbb{R}^4) \mid \sup_{x \in \mathbb{R}^4} |x^\alpha \partial^\beta f(x)| < \infty, \quad \alpha, \beta \in \mathbb{N}_0^4 \}, \quad (7)$$

where $x^\alpha := x_0^{\alpha_0} \dots x_3^{\alpha_3}$ and $\partial^\beta = \frac{\partial^{|\beta|}}{(\partial x^0)^{\beta_0} \dots (\partial x^3)^{\beta_3}}$, $|\beta| = \beta_0 + \dots + \beta_3$.

2. The semi-norms $\|f\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^4} |x^\alpha \partial^\beta f(x)|$ give a notion of convergence in S : $f_n \rightarrow f$ in S if $\|f_n - f\|_{\alpha,\beta} \rightarrow 0$ for all α, β .
3. We say that a linear functional $\varphi : S \rightarrow \mathbb{C}$ is continuous, if for any finite set F of multiindices there is a constant c_F s.t.

$$|\varphi(f)| \leq c_F \sum_{\alpha,\beta \in F} \|f\|_{\alpha,\beta}. \quad (8)$$

(Note that if $f_n \rightarrow f$ in S then $\varphi(f_n) \rightarrow \varphi(f)$). Such continuous functionals are called tempered distributions and form the space S' which is the topological dual of S .

Any measurable, polynomially growing function $x \mapsto \varphi(x)$ defines a tempered distribution via

$$\varphi(f) = \int d^4x \varphi(x) f(x). \quad (9)$$

The notation (9) is often used also if there is no underlying function, e.g. $\delta(f) =: \int \delta(x) f(x) d^4x = f(0)$.

Definition 1.5 *We consider:*

1. A map $S \ni f \mapsto \phi(f) \in \mathcal{L}(D(\phi(f)), \mathcal{H})$.
2. A dense domain $D \subset \mathcal{H}$ s.t. for all $f \in S$
 - $D \subset D(\phi(f)) \cap D(\phi(f)^\dagger)$,
 - $\phi(f) : D \rightarrow D$,
 - $\phi(f)^\dagger : D \rightarrow D$.

We say that (ϕ, D) is an operator valued distribution if for all $\Psi_1, \Psi_2 \in D$ the map

$$S \ni f \mapsto \langle \Psi_1 | \phi(f) \Psi_2 \rangle \in \mathbb{C} \quad (10)$$

is a tempered distribution. We say that (ϕ, D) is Hermitian, if $\phi(f)$ is a Hermitian operator for any real valued $f \in S$.

Note that a posteriori $S \ni f \mapsto \phi(f) \in \mathcal{L}(D, D)$.

1.2.2 Wightman QFT

Definition 1.6 *A theory of one scalar Hermitian Wightman field is given by:*

1. A relativistic quantum theory (\mathcal{H}, U) with a unique vacuum state $\Omega \in \mathcal{H}$.
2. A Hermitian operator-valued distribution (ϕ, D) s.t. $\Omega \in D$ and $U(\Lambda, a)D \subset D$ for all $(\Lambda, a) \in \mathcal{P}_+^\uparrow$ satisfying:

- (a) (*Locality*) $[\phi(f_1), \phi(f_2)] = 0$ if $\text{supp } f_1$ and $\text{supp } f_2$ spacelike separated. (In the sense of weak commutativity on D).
- (b) (*Covariance*) $U(\Lambda, a)\phi(f)U(\Lambda, a)^\dagger = \phi(f_{(\Lambda, a)})$, for all $(\Lambda, a) \in \mathcal{P}_+^\dagger$ and $f \in S$. Here $f_{(\Lambda, a)}(x) = f(\Lambda^{-1}(x - a))$.
- (c) (*Cyclicity of the vacuum*) $\mathcal{D} = \text{Span}\{\phi(f_1) \dots \phi(f_m)\Omega \mid f_1, \dots, f_m \in S, m \in \mathbb{N}_0\}$ is a dense subspace of \mathcal{H} .

The distribution (ϕ, D) is called the *Wightman quantum field*.

Remarks:

1. Operator valued *functions* satisfying the Wightman axioms do not exist (we really need distributions). The physical reason is the uncertainty relation: Measuring ϕ strictly at a point x causes very large fluctuations of energy and momentum, which prevent $\phi(x)$ from being a well defined operator. Such observations were made already in [7], before the theory of distributions was developed.
2. It is possible to choose $D = \mathcal{D}$.

Example: Let \mathcal{H} be the symmetric Fock space, then the energy-momentum operators

$$P^0 = \int \frac{d^3p}{(2\pi)^3 2p^0} p^0 a^\dagger(p)a(p), \quad \vec{P} = \int \frac{d^3p}{(2\pi)^3 2p^0} \vec{p} a^\dagger(p)a(p), \quad (11)$$

where $p_0 = \sqrt{p^2 + m^2}$, satisfy the spectrum condition and generate a unitary representation of translations $U(a) = e^{iP_\mu a^\mu}$. Clearly, $\Omega = |0\rangle$ is the unique vacuum state of this relativistic QM. The Hermitian operator-valued distribution, given in the function notation by

$$\phi_0(x) = \int \frac{d^3p}{(2\pi)^3 2p^0} (e^{ipx} a^\dagger(p) + e^{-ipx} a(p)). \quad (12)$$

is a scalar Hermitian Wightman field.

2 Path integrals

The main references for this section are [5, Chapter 6] [6, Chapter 1].

2.1 Wightman and Schwinger functions

Consider a theory $(\mathcal{H}, U, \Omega, \phi, D)$ of one scalar Wightman field.

- Wightman functions are defined as

$$W_n(x_1, \dots, x_n) = \langle \Omega | \phi(x_1) \dots \phi(x_n) \Omega \rangle. \quad (13)$$

They are tempered distributions.

- Green functions are defined as

$$G_n(x_1, \dots, x_n) = \langle \Omega | T(\phi(x_1) \dots \phi(x_n)) \Omega \rangle \quad (14)$$

Recall that $T\phi(x_1)\phi(x_2) = \theta(x_1^0 - x_2^0)\phi(x_1)\phi(x_2) + \theta(x_2^0 - x_1^0)\phi(x_2)\phi(x_1)$. This multiplication of distributions by a discontinuous function may be ill-defined in the Wightman setting. Approximation of θ by smooth functions may be necessary. Then we obtain tempered distributions.

- Euclidean Green functions (Schwinger functions) are defined as

$$G_{E,n}(x_1, \dots, x_n) = W_n((ix_1^0, \vec{x}_1), \dots, (ix_n^0, \vec{x}_n)). \quad (15)$$

The analytic continuation is justified in the Wightman setting. We obtain real-analytic functions on $\mathbb{R}_{\neq}^{4n} = \{(x_1, \dots, x_n) \mid x_i \neq x_j \ \forall i \neq j\}$, symmetric under the exchange of variables.

The Schwinger functions are central objects of mathematical QFT based on path-integrals. The idea is to express $G_{E,n}$ as moment functions of a measure μ on the space $S'_{\mathbb{R}}$ of real-valued tempered distributions

$$G_{E,n}(x_1, \dots, x_n) = \int_{S'_{\mathbb{R}}} \varphi(x_1) \dots \varphi(x_n) d\mu(\varphi). \quad (16)$$

Today's lecture:

- Measure theory on topological spaces.
- Conditions on $d\mu$ which guarantee that formula (16) really gives Schwinger functions of some Wightman QFT. (Osterwalder-Schrader axioms).
- Remarks on construction of interacting functional measures $d\mu$

2.2 Elements of measure theory

1. Def. We say that X is a topological space, if it comes with a family of subsets $\mathcal{T} = \{O_i\}_{i \in \mathbb{I}}$ of X satisfying the following axioms:

- $\emptyset, X \in \mathcal{T}$,
- $\bigcup_{j \in \mathbb{J}} O_j \in \mathcal{T}$,
- $\bigcap_{j=1}^N O_j \in \mathcal{T}$.

O_i are called the open sets.

2. **Example:** $S'_\mathbb{R}$ is a topological space. In fact, given $\varphi_0 \in S'_\mathbb{R}$, a finite family $J_1, \dots, J_N \in S_\mathbb{R}$ and $\varepsilon_1, \dots, \varepsilon_N > 0$ we can define a *neighbourhood* of φ_0 as follows:

$$\begin{aligned} B(\varphi_0; J_1, \dots, J_N; \varepsilon_1, \dots, \varepsilon_N) \\ := \{ \varphi \in S'_\mathbb{R} \mid |\varphi(J_1) - \varphi_0(J_1)| < \varepsilon_1, \dots, |\varphi(J_N) - \varphi_0(J_N)| < \varepsilon_N \}. \end{aligned} \quad (17)$$

All open sets in $S'_\mathbb{R}$ can be obtained as unions of such neighbourhoods.

3. Def. Let X be a topological space. A family \mathcal{M} of subsets of X is a σ -algebra in X if it has the following properties:

- $X \in \mathcal{M}$,
- $A \in \mathcal{M} \Rightarrow X \setminus A \in \mathcal{M}$,
- $A_n \in \mathcal{M}, n \in \mathbb{N}, \Rightarrow A := \bigcup_{n=1}^\infty A_n \in \mathcal{M}$.

If \mathcal{M} is a σ -algebra in X then X is called a *measurable space* and elements of \mathcal{M} are called *measurable sets*.

4. Def. The Borel σ -algebra is the smallest σ -algebra containing all open sets of X . Its elements are called Borel sets.
5. Def. Let X be a measurable space and Y a topological space. Then a map $f : X \rightarrow Y$ is called *measurable* if for any open $V \subset Y$ the inverse image $f^{-1}(V)$ is a measurable set.
6. Def. A measure is a function $\mu : \mathcal{M} \rightarrow [0, \infty]$ s.t. for any countable family of disjoint sets $A_i \in \mathcal{M}$ we have

$$\mu\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty \mu(A_i). \quad (18)$$

Also, we assume that $\mu(A) < \infty$ for at least one $A \in \mathcal{M}$.

- If $\mu(X) = 1$, we say that μ is a probability measure.
- If μ is defined on the Borel σ -algebra, we call it a Borel measure.

7. We denote by $\mathcal{L}^p(X, d\mu)$, $1 \leq p < \infty$ the space of measurable functions $f : X \rightarrow \mathbb{C}$ s.t.

$$\|f\|_p := \left(\int_X |f(x)|^p d\mu(x) \right)^{1/p} < \infty. \quad (19)$$

We denote by $L^p(X, d\mu)$ the space of equivalence classes of functions from $\mathcal{L}^p(X, d\mu)$ which are equal except at sets of measure zero. The following statements are known as the Riesz-Fisher theorem:

- $L^p(X, d\mu)$ is a Banach space with the norm (19).
- $L^2(X, d\mu)$ is even a Hilbert space w.r.t. $\langle f_1 | f_2 \rangle = \int \bar{f}_1(x) f_2(x) d\mu(x)$.

8. The following theorem allows us to construct measures on $S'_\mathbb{R}$:

Theorem 2.1 (Bochner-Minlos) *Let $Z_E : S_\mathbb{R} \rightarrow \mathbb{C}$ be a map satisfying*

- (a) (Continuity) $Z_E[J_n] \rightarrow Z_E[J]$ if $J_n \rightarrow J$ in $S_\mathbb{R}$
- (b) (Positive definiteness) For any $J_1, \dots, J_N \in S'_\mathbb{R}$, the matrix $A_{i,j} := Z_E[J_i - J_j]$ is positive. This means $z^\dagger A z := \sum_{i,j} \bar{z}_i A_{i,j} z_j \geq 0$ for any $z \in \mathbb{C}^N$.
- (c) (Normalisation) $Z_E[0] = 1$.

Then there exists a unique Borel probability measure μ on $S'_\mathbb{R}$ s.t.

$$Z_E[J] = \int_{S'_\mathbb{R}} e^{i\varphi(J)} d\mu(\varphi) \quad (20)$$

$Z_E[f]$ is called the characteristic function of μ or the (Euclidean) generating functional of the moments of μ . Indeed, formally we have:

$$(-i)^n \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} Z_E[J]|_{J=0} = \int_{S'_\mathbb{R}} \varphi(x_1) \cdots \varphi(x_n) d\mu(\varphi), \quad (21)$$

so the generating functional carries information about all the moments of the measure (cf. (16) above).

9. **Example:** Let $C = \frac{1}{-\Delta + m^2}$, where $\Delta = \frac{\partial^2}{(\partial x^0)^2} + \cdots + \frac{\partial^2}{(\partial x^3)^2}$ is the Laplace operator on \mathbb{R}^4 . For We consider the expectation value of C on $f \in S_\mathbb{R}$:

$$\langle J | C J \rangle := \int d^4 p \bar{\hat{J}}(p) \frac{1}{p^2 + m^2} \hat{J}(p). \quad (22)$$

and set $Z_{E,C}[J] := e^{-\frac{1}{2} \langle J | C J \rangle}$. This map satisfies the assumptions of the Bochner-Minlos theorem and gives a measure $d\mu_C$ on $S'_\mathbb{R}$ called the Gaussian measure with covariance (propagator) C . In the physics notation:

$$\begin{aligned} \int F(\varphi) d\mu_C(\varphi) &= \int F(\varphi) \frac{1}{N_C} e^{-\frac{1}{2} \int d^4 x \varphi(x) (-\Delta + m^2) \varphi(x)} \mathcal{D}[\varphi] \\ &= \int F(\varphi) \frac{1}{N_C} e^{-\frac{1}{2} \int d^4 x (\partial_\mu \varphi(x) \partial^\mu \varphi(x) + m^2 \varphi^2(x))} \mathcal{D}[\varphi], \end{aligned} \quad (23)$$

for any $F \in L^1(S'_\mathbb{R}, d\mu_C)$. Since we chose imaginary time, we have a Gaussian damping factor and not an oscillating factor above. This is the main reason to work in the Euclidean setting.

2.3 Osterwalder-Schrader axioms

Now we formulate conditions, which guarantee that a given measure μ on $S'_\mathbb{R}$ gives rise to a Wightman theory:

Definition 2.2 *We say that a Borel probability measure μ on $S'_\mathbb{R}$ defines an Osterwalder-Schrader QFT if this measure, resp. its generating functional $Z_E : S_\mathbb{R} \rightarrow \mathbb{C}$, satisfies:*

1. (Analyticity) *The function $\mathbb{C}^N \ni (z_1, \dots, z_N) \rightarrow Z_E[\sum_{i=1}^N z_i J_i] \in \mathbb{C}$ is entire analytic for any $J_1, \dots, J_n \in S_\mathbb{R}$.*

Gives existence of Schwinger functions.

2. (Regularity) *For some $1 \leq p \leq 2$, a constant c and all $J \in S_\mathbb{R}$, we have*

$$|Z_E[J]| \leq e^{c(\|J\|_1 + \|J\|_p^p)}. \quad (24)$$

Gives temperedness of the Wightman field.

3. (Euclidean invariance) *$Z_E[J] = Z_E[J_{(R,a)}]$ for all $J \in S_\mathbb{R}$, where $J_{(R,a)}(x) = f(R^{-1}(x - a))$, $R \in SO(4)$, $a \in \mathbb{R}^4$.*

Gives Poincaré covariance of the Wightman theory.

4. (Reflection positivity) *Define:*

- $\theta(x^0, \vec{x}) = (-x^0, \vec{x})$ the Euclidian time reflection.
- $J_\theta(x) := J(\theta^{-1}x) = J(\theta x)$ for $J \in S_\mathbb{R}$.
- $\mathbb{R}_+^4 = \{(x^0, \vec{x}) \mid x^0 > 0\}$

Reflection positivity requires that for functions $J_1, \dots, J_N \in S_\mathbb{R}$, supported in \mathbb{R}_+^4 , the matrix $M_{i,j} := Z_E[J_i - (J_j)_\theta]$ is positive.

Gives positivity of the scalar product in the Hilbert space \mathcal{H} (i.e. $\langle \Psi | \Psi \rangle \geq 0$ for all $\Psi \geq 0$). Also locality and spectrum condition.

5. (Ergodicity) *Define:*

- $J_s(x) = J(x^0 - s, \vec{x})$ for $J \in S_\mathbb{R}$.
- $(T(s)\varphi)(J) = \varphi(J_s)$ for $\varphi \in S'_\mathbb{R}$.

Ergodicity requires that for any function $A \in L^1(S'_{\mathbb{R}}, d\mu)$ and $\varphi_1 \in S'_{\mathbb{R}}$

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t A(T_s \varphi_1) ds = \int_{S'_{\mathbb{R}}} A(\varphi) d\mu(\varphi). \quad (25)$$

Gives the uniqueness of the vacuum.

Theorem 2.3 Let μ be a measure on $S'_{\mathbb{R}}$ satisfying the Osterwalder-Schrader axioms. Then the moment functions

$$G_{E,n}(x_1, \dots, x_n) = \int_{S'_{\mathbb{R}}} \varphi(x_1) \dots \varphi(x_n) d\mu(\varphi) \quad (26)$$

exist and are Schwinger functions of a Wightman QFT.

Remark 2.4 The Gaussian measure $d\mu_C$ from the example above satisfies the Osterwalder-Schrader axioms and gives the (scalar, Hermitian) free field.

Some ideas of the proof: The Hilbert space and the Hamiltonian of the Wightman theory is constructed as follows:

- Def: $\mathcal{E} := L^2(S'_{\mathbb{R}}, d\mu)$.
- Def: $A_J(\varphi) := e^{i\varphi(J)}$ for any $J \in S_{\mathbb{R}}$ and $(\theta A_J)(\varphi) := e^{i\varphi(J_\theta)}$.
- Fact: $\mathcal{E} = \overline{\text{Span}\{A_J | J \in S_{\mathbb{R}}\}}$
- Def: $\mathcal{E}_+ = \overline{\text{Span}\{A_J | J \in S(\mathbb{R}_+^4)_{\mathbb{R}}\}}$ where $S(\mathbb{R}_+^4)_{\mathbb{R}}$ are real Schwartz-class functions supported in \mathbb{R}_+^4 .
- Fact: $\langle A_1 | A_2 \rangle := \int \overline{(\theta A_1)}(\varphi) A_2(\varphi) d\mu(\varphi)$ is a bilinear form on \mathcal{E}_+ , which is positive (i.e. $\langle A | A \rangle \geq 0$) by reflection positivity. Due to the presence of θ it differs from the the scalar product in \mathcal{E} .
- Def: $\mathcal{N} = \{A \in \mathcal{E}_+ | \langle A | A \rangle = 0\}$ and set $\mathcal{H} = (\mathcal{E}_+ / \mathcal{N})^{\text{cpl}}$, where cpl denotes completion. This \mathcal{H} is the Hilbert space of the Wightman theory.
- $T(t) : \mathcal{E}_+ \rightarrow \mathcal{E}_+$ for $t \geq 0$. It gives rise to a semigroup $e^{-tP_0} : \mathcal{H} \rightarrow \mathcal{H}$ with a self-adjoint, positive generator P_0 - the Hamiltonian. Thus $e^{itP_0} : \mathcal{H} \rightarrow \mathcal{H}$ gives unitary time-evolution.

2.4 Interacting measure

Interacting measures are usually constructed by perturbing the Gaussian measure $d\mu_C$. Reflection positivity severely restricts possible perturbations. Essentially, one has to write:

$$d\mu_I(\varphi) = \frac{1}{N} e^{-\int L_{E,I}(\varphi(x)) d^4x} d\mu_C(\varphi), \quad (27)$$

where N is the normalisation constant and $L_{E,I} : \mathbb{R} \rightarrow \mathbb{R}$ some function (the Euclidean interaction Lagrangian). For example $L_{E,I}(\varphi(x)) = \frac{\lambda}{4!} \varphi(x)^4$. But this leads to problems:

- φ is a distribution so $\varphi(x)^4$ in general does not make sense. This *ultraviolet problem* can sometimes be solved by *renormalization*.
- Integral over whole spacetime ill-defined. (But enforced by the translation symmetry).

For ϕ^4 theory in two-dimensional spacetime these problems were overcome and $d\mu_I$ satisfying the Osterwalder-Schrader axioms was constructed. It was also shown that the resulting theory is interacting, i.e. has non-trivial S -matrix. In the next lecture we will discuss the S -matrix in the Wightman setting.

3 Scattering theory

The main reference for this section is [3, Chapter 16].

3.1 Setting

We consider a Wightman theory $(\mathcal{H}, U, \Omega, \phi, D)$. Recall the key properties

1. Covariance: $U(\Lambda, a)\phi(x)U(\Lambda, a)^{-1} = \phi(\Lambda x + a)$,
2. Locality: $[\phi(x), \phi(y)] = 0$ for $(x - y)^2 < 0$,
3. Cyclicity of Ω : Vectors of the form $\phi(f_1) \dots \phi(f_m)\Omega$ span a dense subspace in \mathcal{H} ,

where smearing with test-functions from S in variables x, y is understood in properties 1. and 2. Furthermore $U(a) = e^{iP_\mu a^\mu}$ and $\text{Sp } P \subset \bar{V}_+$. Today we will impose stronger assumptions on the spectrum:

- A.1. The spectrum contains an isolated mass hyperboloid H_m i.e.

$$H_m \subset \text{Sp } P \subset \{0\} \cup H_m \cup G_{\tilde{m}}, \quad (28)$$

where $H_m = \{p \in \mathbb{R}^4 \mid p^0 = \sqrt{\vec{p}^2 + m^2}\}$, $G_{\tilde{m}} = \{p \in \mathbb{R}^4 \mid p^0 \geq \sqrt{\vec{p}^2 + \tilde{m}^2}\}$ for $\tilde{m} > m$. (In other words, the mass-operator $\sqrt{P_\mu P^\mu}$ has an isolated eigenvalue m . Embedded eigenvalues can also be treated [13], but then scattering theory is more difficult).

- A.2. Define the single-particle subspace $\mathcal{H}(H_m)$ as the spectral subspace of H_m . That is, $\mathcal{H}(H_m) = \chi(P)\mathcal{H}$, where $\chi(P)$ is the characteristic function of H_m evaluated at $P = (P_0, P_1, P_2, P_3)$. We assume that U restricted to $\mathcal{H}(H_m)$ is an irreducible representation of \mathcal{P}_+^\uparrow . (One type of particles).

Theorem 3.1 (*Källén-Lehmann representation*). *For a Wightman field ϕ with $\langle \Omega, \phi(x)\Omega \rangle = 0$ we have*

$$\langle \Omega | \phi(x)\phi(y)\Omega \rangle = \int d\rho(M^2)\Delta_+(x - y; M^2), \quad (29)$$

$$\Delta_+(x - y; M^2) := \langle 0 | \phi_0^{(M)}(x)\phi_0^{(M)}(y) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3 2p^0} e^{ip(y-x)}, \quad (30)$$

where $d\rho(M^2)$ is a measure on \mathbb{R}_+ , $p_0 = \sqrt{\vec{p}^2 + M^2}$, $\phi_0^{(M)}$ is the free scalar field of mass M and $|0\rangle$ is the vacuum vector in the Fock space of this free field theory, whereas Ω is the vacuum of the (possibly interacting) Wightman theory. Furthermore, given the structure of the spectrum (28), we have

$$d\rho(M^2) = Z\delta(M^2 - m^2)d(M^2) + d\tilde{\rho}(M^2) \quad (31)$$

where $Z \geq 0$ and $d\tilde{\rho}$ is supported in $[\tilde{m}^2, \infty)$

We assume in the following that:

- A.3. $\langle \Omega, \phi(x)\Omega \rangle = 0$. This is not a restriction, since a shift by a constant $\phi(x) \mapsto \phi(x) + c$ gives a new Wightman field.
- A.4. $Z \neq 0$ to ensure that $\langle \Psi_1 | \phi(x)\Omega \rangle \neq 0$ for some single-particle vector Ψ_1 (i.e. a vector living on H_m). This means that the particle is ‘elementary’ (as opposed to composite) and we do not need polynomials in the field to create it from the vacuum. This assumption can be avoided at a cost of complications.

3.2 Problem and strategy

Take two single-particle states $\Psi_1, \Psi_2 \in \mathcal{H}(H_m)$. We would like to construct vectors $\Psi^{\text{out}}, \Psi^{\text{in}}$ describing outgoing/ incoming configuration of these two single-particle states Ψ_1, Ψ_2 . Mathematically this problem consists in finding two ‘multiplications’

$$\Psi^{\text{out}} = \Psi_1 \overset{\text{out}}{\times} \Psi_2, \quad (32)$$

$$\Psi^{\text{in}} = \Psi_1 \overset{\text{in}}{\times} \Psi_2, \quad (33)$$

which have all the properties of the (symmetrised) tensor product but take values in \mathcal{H} (and not in $\mathcal{H} \otimes \mathcal{H}$). After all, we know from quantum mechanics, that symmetrised tensor products describe configurations of two undistinguishable bosons.

The strategy is suggested by the standard Fock space theory: With the help of the field ϕ we will construct certain ‘time-dependent creation operators’ $t \mapsto A_{1,t}^\dagger$, $t \mapsto A_{2,t}^\dagger$ s.t.

$$\Psi_1 = \lim_{t \rightarrow \pm\infty} A_{1,t}^\dagger \Omega, \quad \Psi_2 = \lim_{t \rightarrow \pm\infty} A_{2,t}^\dagger \Omega. \quad (34)$$

Then we can try to construct

$$\Psi^{\text{out}} = \lim_{t \rightarrow \infty} A_{1,t}^\dagger A_{2,t}^\dagger \Omega, \quad \Psi^{\text{in}} = \lim_{t \rightarrow -\infty} A_{1,t}^\dagger A_{2,t}^\dagger \Omega. \quad (35)$$

Of course analogous consideration applies to n -particle scattering states.

Plan of the remaining part of the lecture:

- Construction of A_t^\dagger .
- Existence of limits in definitions of $\Psi^{\text{out}}, \Psi^{\text{in}}$.
- Wave-operators, S -matrix and the LSZ reduction formula.

3.3 Definition of A_t^\dagger

The operators A_t^\dagger are defined in (41) below. In order to motivate this definition, we state several facts about the free field. It should be kept in mind that we are interested in the interacting field, and the following discussion of the free field is merely a motivating digression.

Recall the definition of the free scalar field:

$$\phi_0(x) = \int \frac{d^3p}{(2\pi)^3 2p^0} (e^{ipx} a^\dagger(p) + e^{-ipx} a(p)). \quad (36)$$

(Here and in the following we reserve the letter p for momenta restricted to the mass-shell i.e. $p = (p^0, \vec{p}) = (\sqrt{\vec{p}^2 + m^2}, \vec{p})$. For other momenta I will use q). There are two ways to extract a^\dagger out of ϕ_0 :

1. Use the formula from the lecture:

$$a^\dagger(p) = i \int d^3x \phi_0(x) \overleftrightarrow{\partial}_0 e^{-ipx} \quad (37)$$

Since $a^\dagger(p)$ is not a well-defined operator (only an operator valued distribution) we will smear both sides of this equality with a test-function. For this purpose we define for any $f \in C_0^\infty(\mathbb{R}^4)$

$$a^\dagger(f) := \int \frac{d^3p}{(2\pi)^3 2p^0} a^\dagger(p) f(p), \quad f_m(x) = \int \frac{d^3p}{(2\pi)^3 2p^0} e^{-ipx} f(p), \quad (38)$$

where the latter is a positive-energy solution of the KG equation, that is $(\square + m^2)f_m(x) = 0$. We get

$$a^\dagger(f) = i \int d^3x \phi_0(x) \overleftrightarrow{\partial}_0 f_m(x). \quad (39)$$

2. Pick a function $h \in S$ s.t. $\text{supp } \widehat{h}$ is compact and $\text{supp } \widehat{h} \cap \text{Sp } P \subset H_m$. Then

$$\phi_0(h) = (2\pi)^2 a^\dagger(\widehat{h}), \quad \text{where } \widehat{h}(q) = \frac{1}{(2\pi)^2} \int e^{iqx} h(x) d^4x. \quad (40)$$

Now we come back to our (possibly) interacting Wightman field ϕ and perform both operations discussed above to obtain the ‘time dependent creation operator’

$$A_t^\dagger := i \int d^3x \phi(h)(t, \vec{x}) \overleftrightarrow{\partial}_0 f_m(t, \vec{x}), \quad (41)$$

where $\phi(h)(t, \vec{x}) := U(t, \vec{x}) \phi(h) U(t, \vec{x})^\dagger = \phi(h_{(t, \vec{x})})$.

3.4 Construction of scattering states

Theorem 3.2 (*Haag-Ruelle*) For f_1, \dots, f_n with disjoint supports, the following limits exist

$$\Psi_n^{\text{out}} = \lim_{t \rightarrow \infty} A_{1,t}^\dagger \dots A_{n,t}^\dagger \Omega, \quad (42)$$

$$\Psi_n^{\text{in}} = \lim_{t \rightarrow -\infty} A_{1,t}^\dagger \dots A_{n,t}^\dagger \Omega \quad (43)$$

and define outgoing/incoming scattering states.

Proof. For $n = 1$ the expression

$$A_{1,t}^\dagger \Omega = i \int d^3x (\phi(h)(t, \vec{x}) \Omega) \overset{\leftrightarrow}{\partial}_0 f_m(t, \vec{x}) \quad (44)$$

is independent of t and thus $\lim_{t \pm \infty} A_{1,t}^\dagger (f_1) \Omega$ (trivially) exist. Moreover, it is a single-particle state. Justification:

- $x \mapsto \phi(h)(x) \Omega$ is a solution of the KG equation. This can be shown using the Källén-Lehmann representation and the support property of \widehat{h} to eliminate the contribution from $d\tilde{\rho}$. Assumption A.1. enters here. (Howework).
- For any two solutions g_1, g_2 of the KG equation $\int d^3x g_1(t, \vec{x}) \overset{\leftrightarrow}{\partial}_0 g_2(t, \vec{x})$ is independent of t .
- We have $i[P_\mu, \phi(h)(x)] = (\frac{\partial}{\partial x^\mu}) \phi(h)(x)$. Since $P_\mu \Omega = 0$, we can write

$$\begin{aligned} P^2 \phi(h)(x) \Omega &= P_\mu P^\mu \phi(h)(x) \Omega = -i[P_\mu, i[P^\mu, \phi(h)(x)]] \Omega \\ &= -\square_x \phi(h)(x) \Omega = m^2 \phi(h)(x) \Omega, \end{aligned} \quad (45)$$

where in the last step we used the first item above. Hence $\phi(h)(x) \Omega$ are single-particle states of mass m .

For $n = 2$ we set $\Psi_t := A_{1,t}^\dagger A_{2,t}^\dagger \Omega$ and try to verify the Cauchy criterion:

$$\|\Psi_{t_2} - \Psi_{t_1}\| = \left\| \int_{t_1}^{t_2} \partial_\tau \Psi_\tau d\tau \right\| \leq \int_{t_1}^{t_2} \|\partial_\tau \Psi_\tau\| d\tau. \quad (46)$$

If we manage to show that $\|\partial_\tau \Psi_\tau\| \leq c/\tau^{1+\eta}$ for some $\eta > 0$ then the Cauchy criterion will be satisfied as we will have

$$\|\Psi_{t_2} - \Psi_{t_1}\| \leq c \left| \frac{1}{t_1^\eta} - \frac{1}{t_2^\eta} \right|. \quad (47)$$

(Note that we use the completeness of \mathcal{H} here i.e. the property that any Cauchy sequence converges).

Thus we study $\partial_\tau \Psi_\tau$. The Leibniz rule gives

$$\begin{aligned}\partial_\tau \Psi_\tau &= (\partial_\tau A_{1,\tau}^\dagger) A_{2,\tau}^\dagger \Omega + A_{1,\tau}^\dagger (\partial_\tau A_{2,\tau}^\dagger) \Omega \\ &= [(\partial_\tau A_{1,\tau}^\dagger), A_{2,\tau}^\dagger] \Omega + A_{2,\tau}^\dagger (\partial_\tau A_{1,\tau}^\dagger) \Omega + A_{1,\tau}^\dagger (\partial_\tau A_{2,\tau}^\dagger) \Omega.\end{aligned}\quad (48)$$

Since $(\partial_\tau A_{i,\tau}^\dagger) \Omega = 0$ by the first part of the proof, only the term with the commutator above is non-zero. To analyze it, we need some information about KG wave-packets:

- Def. For the KG wave-packet $f_{i,m}$ we define the velocity support as

$$V_i = \left\{ \frac{\vec{p}}{p^0} \mid p \in \text{supp } f_i \right\} \quad (49)$$

and let V_i^δ be slightly larger sets.

- Fact. For any $N \in \mathbb{N}$ we can find a c_N s.t.

$$|f_{i,m}(\tau, \vec{x})| \leq \frac{c_N}{\tau^N} \text{ for } \frac{\vec{x}}{\tau} \notin V_i^\delta. \quad (50)$$

Due to (50), the contributions to $\|[(\partial_\tau A_{1,\tau}^\dagger), A_{2,\tau}^\dagger] \Omega\|$ coming from the part of the integration region in (41) where $\frac{\vec{x}}{\tau} \notin V_i^\delta$, are rapidly vanishing with τ . So we only have to worry about the dominant parts:

$$A_{i,t}^{\dagger(D)} := i \int_{\frac{\vec{x}}{t} \in V_i^\delta} d^3x \phi(h)(t, \vec{x}) \overset{\leftrightarrow}{\partial}_0 f_{i,m}(t, \vec{x}). \quad (51)$$

Since V_1^δ, V_2^δ are disjoint, the Wightman axiom of locality gives for sufficiently large τ .

$$\|[(\partial_\tau A_{1,\tau}^{\dagger(D)}), A_{2,\tau}^{\dagger(D)}] \Omega\| \leq \frac{c_N}{\tau^N}. \quad (52)$$

This concludes the proof. \square

3.5 Wave operators, scattering matrix, LSZ reduction

In the following we choose h s.t. $\widehat{h}(p)f(p) = (2\pi)^{-2}Z^{-1/2}f(p)$. This can be done, since f has compact support. After this fine-tuning, exploiting assumptions A.2, A.3, A.4 one obtains the following simple formula for scalar products of scattering states:

Theorem 3.3 (*Haag-Ruelle*) *Let $\Psi_n^{\text{out}}, (\Psi_{n'}^{\prime})^{\text{out}}$ be as in the previous theorem. Then their scalar products can be computed as if these were vectors on the Fock space:*

$$\langle \Psi_n^{\text{out}} | (\Psi_{n'}^{\prime})^{\text{out}} \rangle = \langle 0 | a(f_n) \dots a(f_1) a^\dagger(f'_1) \dots a^\dagger(f'_{n'}) | 0 \rangle \quad (53)$$

and analogously for incoming states.

Let \mathcal{F} be the symmetric Fock space. (This is not the Hilbert space of our Wightman theory, but merely an auxiliary object needed to define the wave-operators). We define the outgoing wave-operator $W^{\text{out}} : \mathcal{F} \rightarrow \mathcal{H}$ as

$$W^{\text{out}}(a^\dagger(f_1) \dots a^\dagger(f_n)|0\rangle) = \lim_{t \rightarrow \infty} A_{1,t}^\dagger \dots A_{n,t}^\dagger \Omega. \quad (54)$$

By Theorem 3.3 it is an isometry i.e. $(W^{\text{out}})^\dagger W^{\text{out}} = I$. If it is also a unitary i.e. $\text{Ran } W^{\text{out}} = \mathcal{H}$ then we say that the theory is *asymptotically complete* that is every vector in \mathcal{H} can be interpreted as a collection of particles from $\mathcal{H}(H_m)$. This property does not follow from Wightman axioms (there are counterexamples) and it is actually not always expected on physical grounds. For a more thorough discussion of asymptotic completeness we refer to [12].

The incoming wave-operator $W^{\text{in}} : \mathcal{F} \rightarrow \mathcal{H}$ is defined by taking the limit $t \rightarrow -\infty$ in (54). The scattering matrix $\hat{S} : \mathcal{F} \rightarrow \mathcal{F}$ is given by¹

$$\hat{S} = (W^{\text{out}})^\dagger W^{\text{in}}. \quad (55)$$

If $\hat{S} \neq I$ we say that a theory is interacting. If $\text{Ran } W^{\text{out}} = \text{Ran } W^{\text{in}}$, then \hat{S} is a unitary (even without asymptotic completeness).

Corollary 3.4 (*LSZ reduction*) [8] For $f_1, \dots, f_\ell, g_1, \dots, g_n \in S$ with mutually disjoint supports, we have

$$\begin{aligned} \langle 0|a(f_1) \dots a(f_\ell) \hat{S} a^\dagger(g_1) \dots a^\dagger(g_n)|0\rangle &= \int \frac{d^3 k_1}{(2\pi)^3 2k_1^0} \dots \frac{d^3 p_n}{(2\pi)^3 2p_n^0} f_1(k_1) \dots g_n(p_n) \times \\ &\quad \times \frac{(-i)^{n+\ell}}{(\sqrt{Z})^{n+\ell}} \prod_{i=1}^{\ell} (k_i^2 - m^2) \prod_{j=1}^n (p_j^2 - m^2) \times \\ &\quad \times \int d^4 x_1 \dots d^4 x_\ell d^4 y_1 \dots d^4 y_n e^{i \sum_{i=1}^{\ell} k_i x_i - i \sum_{j=1}^n p_j y_j} \times \\ &\quad \times \langle \Omega | T(\phi(x_\ell) \dots \phi(x_1) \phi(y_1) \dots \phi(y_n)) \Omega \rangle, \end{aligned}$$

where T is the time-ordered product (which needs to be regularized in the Wightman setting).

By analytic continuation one can relate the Green functions to Schwinger functions. The latter can be studied using path integrals as explained in the previous lecture. This led to a proof that for ϕ^4 in 2-dimensional spacetime $\hat{S} \neq I$ [9]. It is a big open problem if there is a Wightman theory in 4-dimensional spacetime with $\hat{S} \neq I$.

¹The notation \hat{S} is used to avoid confusion with the Schwartz class S .

4 Renormalization

Main references for this section are [10, 11].

4.1 Introductory remarks

Consider a Wightman theory $(\mathcal{H}, U, \Omega, \phi_B, D)$ as in the previous lecture and suppose we want to describe a collision of several particles: The LSZ formula gives:

$$\begin{aligned} \langle 0|a(k_1)\dots a(k_\ell)\hat{S}a^\dagger(p_1)\dots a^\dagger(p_n)|0\rangle &= \frac{(-i)^{n+\ell}}{(\sqrt{Z})^{n+\ell}} \prod_{i=1}^{\ell}(k_i^2 - m^2) \prod_{j=1}^n (p_j^2 - m^2) \times \\ &\times \int d^4x_1\dots d^4x_\ell d^4y_1\dots d^4y_n e^{i\sum_{i=1}^{\ell} k_i x_i - i\sum_{j=1}^n p_j y_j} \times \\ &\times \langle \Omega|T(\phi_B(x_\ell)\dots \phi_B(x_1)\phi_B(y_1)\dots \phi_B(y_n))\Omega\rangle, \end{aligned}$$

The ‘renormalized’ field $\phi := Z^{-1/2}\phi_B$ is again a Wightman field. To compute the S -matrix of $(\mathcal{H}, U, \Omega, \phi, D)$ we drop Z and the index B on the r.h.s. of the formula above.

Possibly after regularizing the time-ordered product, we can express the Green functions of ϕ above by the Wightman functions and then analytically continue to Schwinger functions $G_{E,n}$. If the theory satisfies also Osterwalder-Schrader axioms, we have

$$G_{E,n}(x_1, \dots, x_n) = \frac{\delta}{\delta J(x_1)} \dots \frac{\delta}{\delta J(x_n)} Z_E[J]|_{J=0}, \quad Z_E[J] = \int_{S_{\mathbb{R}}} e^{\varphi(J)} d\mu(\varphi), \quad (56)$$

for some Borel measure $d\mu$ and $J \in S_{\mathbb{R}}$. (We changed $e^{i\varphi(J)}$ to $e^{\varphi(J)}$, where $J \in S_{\mathbb{R}}$, making use of the Osterwalder-Schrader axiom of analyticity).

Today we will attempt to construct this measure in $\lambda\phi^4$ theory. This endeavor will not be completely successful. In the end we will obtain $G_{E,n}$ as power series in the coupling constant λ whose convergence we will not control. Thereby we will abandon the Wightman/Osterwalder-Schrader setting and delve into perturbative QFT. Strictly speaking, we will also abandon the realm of quantum theories, as there will be no underlying Hilbert space and thus no control that transition probabilities of physical processes take values between zero and one. On the positive side, the finiteness of individual terms in this expansion will be an interesting problem in the theory of differential equations.

4.2 From action to generating functional

Consider the Euclidean action written in terms of the bare (unphysical) quantities:

$$S_E[\varphi_B] = \int d^4x \left(\frac{1}{2} \partial_\mu \varphi_B \partial^\mu \varphi_B + \frac{1}{2} m_B^2 \varphi_B^2 + \frac{\lambda_B}{4!} \varphi_B^4 \right). \quad (57)$$

Introduce the physical parameters via $\varphi_B = \sqrt{Z}\varphi$, $m_B = Z_m m$, $\lambda_B = Z_\lambda \lambda$. This gives

$$\begin{aligned} S_E[\varphi] &= \int d^4x \frac{1}{2} (\partial_\mu \varphi \partial^\mu \varphi + m^2 \varphi^2) \\ &+ \int d^4x \left\{ \frac{1}{2} (Z - 1) \partial_\mu \varphi \partial^\mu \varphi + \frac{1}{2} (Z_m^2 Z - 1) m^2 \varphi^2 + \frac{Z_\lambda Z^2}{4!} \lambda \varphi^4 \right\} \\ &= S_{E,0}[\varphi] + S_{E,I}[\varphi]. \end{aligned} \quad (58)$$

Let us start with the free action $S_{E,0}$. The corresponding generating functional is

$$Z_{E,0}[J] = e^{\frac{1}{2} \int d^4p \bar{J}(p) C(p) \hat{J}(p)} = \int_{S'_\mathbb{R}} e^{\varphi(J)} d\mu_C(\varphi), \quad (59)$$

$$d\mu_C(\varphi) = \text{''} \frac{1}{N_C} e^{-S_{E,0}[\varphi]} \mathcal{D}[\varphi] \text{''}, \quad (60)$$

where $C(p) = \frac{1}{p^2 + m^2}$ and $d\mu_C$ is the corresponding Gaussian measure given formally by (60). The candidate generating functional for the interacting theory is

$$Z_E^{\text{cand}}[J] = \frac{1}{N} \int_{S'_\mathbb{R}} e^{\varphi(J)} e^{-S_{E,I}[\varphi]} d\mu_C(\varphi), \quad (61)$$

where N is chosen so that $Z_E^{\text{cand}}[0] = 1$. Problems:

- $d\mu_C$ is supported on distributions so $\varphi(x)^4$ appearing in $S_{E,I}[\varphi]$ is ill-defined (UV problem).
- Integral over spacetime volume in $S_{E,I}[\varphi]$ ill-defined (IR problem).

4.3 Regularized generating functional

To make sense out of (61) we need regularization: We set

$$C_\Lambda^{\Lambda_0}(p) := \frac{1}{p^2 + m^2} \left(e^{-\frac{p^2 + m^2}{\Lambda_0^2}} - e^{-\frac{p^2 + m^2}{\Lambda^2}} \right). \quad (62)$$

Here Λ_0 is the actual UV cut-off. $0 \leq \Lambda \leq \Lambda_0$ is an auxiliary cut-off which will be needed for technical reasons. Thus $C_\Lambda^{\Lambda_0}$ is essentially supported in $\Lambda \leq p^2 \leq \Lambda_0$.

Lemma 4.1 *$d\mu_{C_\Lambda^{\Lambda_0}}$ is supported on smooth functions.*

Thus the following regularized generating functional is meaningful

$$Z_E^{\Lambda, \Lambda_0}[J] = \frac{1}{N} \int e^{\varphi(J)} e^{-S_{E,I,(V)}^{\Lambda_0}[\varphi]} d\mu_{C_\Lambda^{\Lambda_0}}(\varphi), \quad (63)$$

where we also introduced a finite volume V . Of course, when we try to take the limit $\Lambda_0 \rightarrow \infty$, we will get back divergent expressions. The idea of renormalization

is to absorb these divergencies into the parameters in $S_{E,I,(V)}^{\Lambda_0}$ (therefore we added a superscript Λ_0). For simplicity, we rename the coefficients from (58) as follows:

$$S_{E,I,(V)}^{\Lambda_0}[\varphi] = \int_V d^4x (a^{\Lambda_0} \partial_\mu \varphi \partial^\mu \varphi + b^{\Lambda_0} \varphi^2 + c^{\Lambda_0} \lambda \varphi^4). \quad (64)$$

We list the following facts:

- Z_E^{Λ, Λ_0} generates (regularized) Schwinger functions $G_{E,n}^{\Lambda, \Lambda_0}$.
- $-\log(NZ_E^{\Lambda, \Lambda_0})$ generates (regularized) connected Schwinger functions $G_{E,c,n}^{\Lambda, \Lambda_0}$.
- $\Sigma^{\Lambda, \Lambda_0}[J] := -\log(NZ_E^{\Lambda, \Lambda_0}[(C_\Lambda^{\Lambda_0})^{-1}J]) + \frac{1}{2}\langle J|(C_\Lambda^{\Lambda_0})^{-1}J\rangle$ generates (regularized) connected amputated Schwinger functions with subtracted zero-order contribution $G_{E,r \geq 1, a, c, n}^{\Lambda, \Lambda_0}$.

Given energy-momentum conservation, it is convenient to set

$$\begin{aligned} S_n^{\Lambda, \Lambda_0}(p_1, \dots, p_{n-1})(2\pi)^4 \delta(p_1 + \dots + p_n) \\ := \int e^{i(p_1 x_1 + \dots + p_n x_n)} G_{E, r \geq 1, a, c, n}^{\Lambda, \Lambda_0}(x_1, \dots, x_n) d^4 x_1 \dots d^4 x_n \end{aligned} \quad (65)$$

4.4 The problem of perturbative renormalizability

The renormalization of ϕ^4 in 4 dimensions is only understood perturbatively. This means we treat S_n^{Λ, Λ_0} as a formal power series

$$S_n^{\Lambda, \Lambda_0} = \sum_{r \geq 1} \lambda^r S_{r,n}^{\Lambda, \Lambda_0}, \quad (66)$$

i.e. a series whose convergence we do not control. (Order by order the limit $V \rightarrow \mathbb{R}^4$ can be taken, so we will not discuss it anymore). Similarly, we treat the coefficients from the interaction Lagrangian as formal power series:

$$a^{\Lambda_0} = \sum_{r \geq 1} \lambda^r a_r^{\Lambda_0}, \quad b^{\Lambda_0} = \sum_{r \geq 1} \lambda^r b_r^{\Lambda_0}, \quad c^{\Lambda_0} = \sum_{r \geq 1} \lambda^r c_r^{\Lambda_0}. \quad (67)$$

Furthermore, we impose the following BPHZ renormalization conditions (RC)

$$S_{r,4}^{0, \Lambda_0}(0) = \delta_{r,1}, \quad S_{r,2}^{0, \Lambda_0}(0) = 0, \quad \partial_{p^2} S_{r,2}^{0, \Lambda_0}(0) = 0, \quad (68)$$

which fix the physical values of the parameters².

Theorem 4.2 (*Perturbative renormalizability*) *There are such $\{a_r^{\Lambda_0}, b_r^{\Lambda_0}, c_r^{\Lambda_0}\}_{\Lambda_0 \geq 0}$ that the limits*

$$S_{r,n}(\underline{p}) := \lim_{\Lambda_0 \rightarrow \infty} S_{r,n}^{0, \Lambda_0}(\underline{p}), \quad \underline{p} = (p_1, \dots, p_{n-1}) \quad (69)$$

exist and are finite, and the renormalisation conditions (68) hold.

²The physical meaning of our RC is not so direct, since we are in the Euclidean setting. Before substituting our n-point functions to the LSZ formula for the S -matrix, they have to be analytically continued to the real time and transferred to the on-shell renormalization scheme.

Example: Consider the leading contribution to the two-point and four-point function:

$$\begin{aligned}
S_{1,2}^{0,\Lambda_0}(p) &= \dots\dots O\dots\dots + \dots\dots x\dots\dots \\
&= 12c_1^{\Lambda_0} \int \frac{d^4q}{(2\pi)^4} C_0^{\Lambda_0}(q) + 2(a_1^{\Lambda_0} p^2 + b_1^{\Lambda_0}), \\
&= 12c_1^{\Lambda_0} \underbrace{\int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + m^2} e^{-\frac{(q^2+m^2)}{\Lambda_0^2}}}_{I^{\Lambda_0}} + 2(a_1^{\Lambda_0} p^2 + b_1^{\Lambda_0}), \quad (70)
\end{aligned}$$

$$S_{1,4}^{0,\Lambda_0}(p_1, p_2, p_3) = 4!c_1^{\Lambda_0}. \quad (71)$$

For any finite Λ_0 the integral above is convergent, but it diverges for $\Lambda_0 \rightarrow \infty$. We have to choose the behavior of $a_1^{\Lambda_0}, b_1^{\Lambda_0}, c_1^{\Lambda_0}$ s.t. this divergence is compensated and the RC are satisfied. We get from the three RC conditions, respectively,

$$4!c_1^{\Lambda_0} = 1, \quad 12c_1^{\Lambda_0} I^{\Lambda_0} + 2b_1^{\Lambda_0} = 0, \quad 2a_1^{\Lambda_0} = 0. \quad (72)$$

Hence, $c_1^{\Lambda_0} = 1/4!$, $b_1^{\Lambda_0} = -(1/4)I^{\Lambda_0}$, $a_1^{\Lambda_0} = 0$ and thus $b_1^{\Lambda_0}$ absorbs the divergence of the integral I^{Λ_0} .

4.5 Flow equations

Properties of $\Sigma^{\Lambda,\Lambda_0}[J] := -\log(NZ_E^{\Lambda,\Lambda_0}[(C_\Lambda^{\Lambda_0})^{-1}J]) + \frac{1}{2}\langle J|(C_\Lambda^{\Lambda_0})^{-1}J\rangle$:

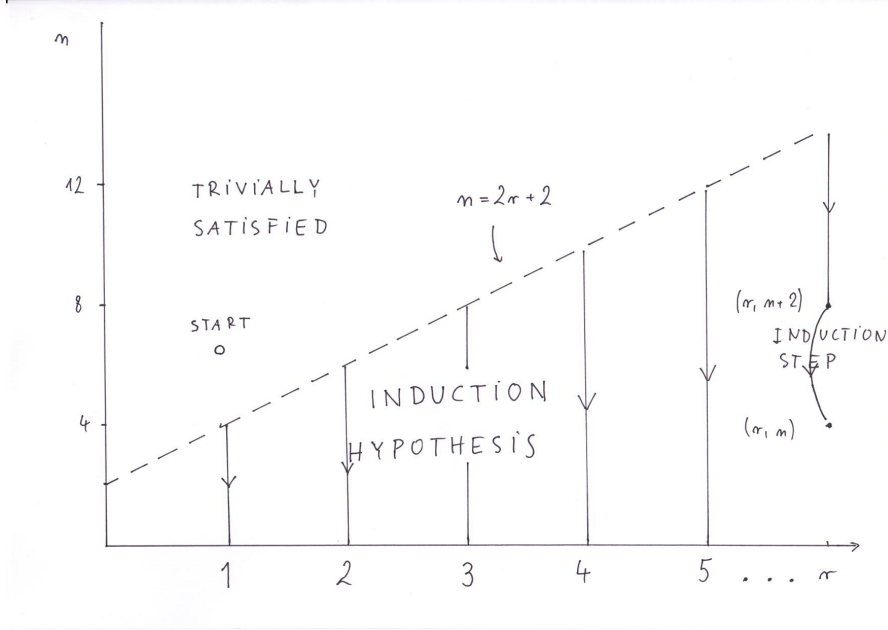
- $e^{-\Sigma^{\Lambda,\Lambda_0}[J]} = \int d\mu_{C_\Lambda^{\Lambda_0}}(\varphi) e^{-S_{E,I}^{\Lambda_0}[J+\varphi]}$.
- $\lim_{\Lambda \rightarrow \Lambda_0} \Sigma^{\Lambda,\Lambda_0}[J] = S_{E,I}^{\Lambda_0}[J]$ since $\lim_{\Lambda \rightarrow \Lambda_0} C_\Lambda^{\Lambda_0}(p) = 0$ and therefore, formally, $\lim_{\Lambda \rightarrow \Lambda_0} d\mu_{C_\Lambda^{\Lambda_0}}(\varphi) = \delta(\varphi)\mathcal{D}[\varphi]$.
- $\partial_\Lambda(e^{-\Sigma^{\Lambda,\Lambda_0}[J]}) = \frac{1}{2}\langle \frac{\delta}{\delta J} | \partial_\Lambda C_\Lambda^{\Lambda_0} \frac{\delta}{\delta J} \rangle e^{-\Sigma^{\Lambda,\Lambda_0}[J]}$.

This gives the following equation for $S_{r,n}^{\Lambda,\Lambda_0}(\underline{p}) := S_{r,n}^{\Lambda,\Lambda_0}(p_1, \dots, p_{n-1})$

$$\begin{aligned}
\partial_\Lambda S_{r,n}^{\Lambda,\Lambda_0}(\underline{p}) &= \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} (\partial_\Lambda C_\Lambda^{\Lambda_0})(p) S_{r,n+2}^{\Lambda,\Lambda_0}(p_1, \dots, p_{n-1}, p, -p) \\
&\quad - \frac{1}{2} \sum_{\substack{r'+r''=r \\ n'+n''=n+2}} \binom{n}{n'-1} [S_{r',n'}^{\Lambda,\Lambda_0}(p_1, \dots, p_{n'-1}) (\partial_\Lambda C_\Lambda^{\Lambda_0})(q) S_{r'',n''}^{\Lambda,\Lambda_0}(-q, p_{n'}, \dots, p_{n-1})]_{\text{sym}}, \quad (73)
\end{aligned}$$

where $q = -p_1 - \dots - p_{n'-1}$ and sym denotes the symmetrisation in p_1, \dots, p_{n-1} . Boundary conditions:

$$(a) \quad \Lambda = 0 : S_{r,4}^{0,\Lambda_0}(0) = \delta_{r,1}, \quad S_{r,2}^{0,\Lambda_0}(0) = 0, \quad \partial_{p^2} S_{r,2}^{0,\Lambda_0}(0) = 0 \text{ (RC)}.$$



(b) $\Lambda = \Lambda_0 : \partial^w S_{r,n}^{\Lambda_0, \Lambda_0}(\underline{p}) = 0$ for $n + |w| \geq 5$. (By second bullet above).

With the help of the above equation one proves the following theorem:

Theorem 4.3 *The following estimate holds*

$$|\partial^w S_{r,n}^{\Lambda, \Lambda_0}(\underline{p})| \leq \begin{cases} P_1(|p|) & \text{for } 0 \leq \Lambda \leq 1, \\ \Lambda^{4-n-w} P_2(\log \Lambda) P_3\left(\frac{|p|}{\Lambda}\right) & \text{for } 1 \leq \Lambda \leq \Lambda_0 \end{cases},$$

where P_i are some polynomials independent of p, Λ, Λ_0 , but depending on n, r, w .

In particular this shows that $S_{r,n}^{\Lambda, \Lambda_0}(\underline{p})$ stays bounded if $\Lambda = 0$ and $\Lambda_0 \rightarrow \infty$. With more effort, one also shows convergence as $\Lambda_0 \rightarrow \infty$, required by the renormalizability property (69).

4.6 Outline of the proof of Theorem 4.3

Observation³:

$$S_{r,n}^{\Lambda, \Lambda_0} \equiv 0 \text{ for } n > 2r + 2. \quad (74)$$

Given this, the structure of the flow equation suggests the inductive scheme as in the figure: Indeed, we can start the induction in the region where (74) holds and therefore the estimate from Theorem 4.3 is satisfied. We suppose the estimate holds for

³Indeed, without assuming connectedness we clearly have vanishing of these functions for $n > 4r$. Now $(r - 2)$ vertices need to use at least two lines to keep the diagram connected. 2 vertices need to spend only one line. So altogether $n > 4r - 2(r - 2) - 2$.

- (r, n_1) for $n_1 \geq n + 2$.
- (r_2, n_2) for $r_2 < r$ and any n_2 .

Since only $S_{r', n'}^{\Lambda, \Lambda_0}$ as listed above appear on the r.h.s. of the flow equation (73), we can apply the estimate to this r.h.s. Then (after quite some work) the flow equation gives the required estimate on the $S_{r, n}^{\Lambda, \Lambda_0}$, which appears on the l.h.s. of the flow equation (73).

5 Symmetries I

Symmetries in physics are described by groups. We recall some definitions and facts from the theory of groups and their representations following [14, Chapter 1], [15, Chapter 1], [16].

5.1 Groups

1. Def. A group is a set G with an operation $\cdot : G \times G \rightarrow G$ s.t.
 - $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$ for all $g_1, g_2, g_3 \in G$,
 - There exists $e \in G$ s.t. $g \cdot e = e \cdot g = g$ for all $g \in G$,
 - For any $g \in G$ there exists $g^{-1} \in G$ s.t. $g \cdot g^{-1} = e$.
2. Examples:
 - $\mathbb{Z}_2 = \{1, -1\}$ is the group of parity transformations.
 - Let V be a vector space over the field \mathbb{K} (\mathbb{R} or \mathbb{C}). Then $GL(V)$ denotes the group of all invertible linear mappings $V \rightarrow V$.
 - For $V = \mathbb{K}^n$ we write $GL(n, \mathbb{K}) := GL(V)$. This is the group of invertible $n \times n$ matrices with entries in \mathbb{K} .
 - $SO(3) = \{R \in GL(3, \mathbb{R}) \mid R^T R = I, \det R = 1\}$ - the group of rotations.
 - $SU(2) = \{U \in GL(2, \mathbb{C}) \mid U^\dagger U = I, \det U = 1\}$ - special unitary group.
3. Let G, \hat{G} be groups. Then $H : G \rightarrow \hat{G}$ is a group homomorphism if

$$H(g_1 g_2) = H(g_1) H(g_2) \text{ for any } g_1, g_2 \in G. \quad (75)$$

If H is in addition a bijection then it is called an isomorphism.

(For Lie groups, which we discuss below, homomorphisms are required to be smooth and isomorphisms should also have smooth inverse).

5.2 Lie groups

1. Def. G is a Lie group if it is a smooth real manifold and the group operation and taking the inverse are smooth maps. The dimension of G is the dimension of this manifold.
2. Def. A set M is an n -dimensional smooth manifold if the following hold:
 - It is a Hausdorff topological space. (Distinct points have non-overlapping neighbourhoods, unique limits).
 - There is an *open cover* i.e. a family of open sets $U_\alpha \subset \mathcal{M}$, $\alpha \in \mathcal{I}$, s.t. $\bigcup_{\alpha \in \mathcal{I}} U_\alpha = M$.

- There is an *atlas* $\mathcal{A}(M) := \{ \eta_\alpha : U_\alpha \rightarrow O_\alpha \mid \alpha \in I \}$ given by some family of open sets $O_\alpha \subset \mathbb{R}^n$ and charts η_α , which are homeomorphisms. (A homeomorphism is a continuous bijection whose inverse is also continuous).
 - Let $O_{\alpha,\beta} := \eta_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$ and let $O_{\beta,\alpha} = \eta_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$. Then $\eta_\alpha \circ \eta_\beta^{-1} : O_{\alpha,\beta} \rightarrow O_{\beta,\alpha}$ is smooth.
3. Def. A map $F : M \rightarrow \hat{M}$ between two manifolds is smooth if all the maps $\hat{\eta}_\beta \circ F \circ \eta_\alpha^{-1} : O_\alpha \rightarrow \hat{O}_\beta$ are smooth whenever well-defined. It is called a diffeomorphism if it is a bijection and the inverse is also smooth. We denote by $C^\infty(M)$ the space of smooth maps $M \rightarrow \mathbb{R}$.
4. Def. A smooth map $\mathbb{R} \times M \ni (t, x) \mapsto \gamma_t(x) \in M$, is called a *flow* (of a vector field) if

$$\gamma_0 = \text{id}_M, \quad \gamma_s \circ \gamma_t = \gamma_{s+t} \text{ for } t, s \in \mathbb{R}. \quad (76)$$

5. Def. A vector field $X : C^\infty(M) \rightarrow C^\infty(M)$ with the flow γ is given by

$$X(f) = \frac{d}{dt} f \circ \gamma_t|_{t=0}, \quad f \in C^\infty(M). \quad (77)$$

6. Fact. Given two vector fields X, Y , the commutator $[X, Y](f) := X(Y(f)) - Y(X(f))$ is again a vector field.
7. Def. A tangent vector at point $x \in M$ is a map $X_x : C^\infty(M) \rightarrow \mathbb{R}$ given by $X_x(f) = \frac{d}{dt} f \circ \gamma_t(x)|_{t=0}$. We denote by $T_x M$ the space of all tangent vectors at x (for different flows).

Example: Let $M = \mathbb{R}^n$ and $\gamma_t = (\gamma_t^1, \dots, \gamma_t^n)$ be a flow. Then

$$X_x(f) = \sum_{i=1}^n \frac{d}{dt} \gamma_t^i(x)|_{t=0} \frac{\partial f}{\partial x^i}(x). \quad (78)$$

If one ‘forgets’ f and thinks about $\frac{\partial}{\partial x^i}$ as basis vectors then the expression $X_x = \sum_{i=1}^n \frac{d}{dt} \gamma_t^i(x)|_{t=0} \frac{\partial}{\partial x^i}$ is clearly the tangent vector to $t \rightarrow \gamma_t(x)$ at x .

8. Def. (Transport of a vector field) Let $F : M \rightarrow \hat{M}$ be a diffeomorphism and X a vector field on M given by a flow $\{\gamma_t\}_{t \in \mathbb{R}}$. Then $F_* X$ is a vector field on \hat{M} given by $F \circ \gamma_t \circ F^{-1}$. That is

$$(F_* X)(\hat{f}) = \frac{d}{dt} \hat{f} \circ F \circ \gamma_t \circ F^{-1}|_{t=0}, \quad \hat{f} \in C^\infty(\hat{M}). \quad (79)$$

9. Now we can introduce the Lie algebra G' of a Lie group G .

- Def. Let G be a Lie group and $g \in G$. Then, the left-multiplication $L_g : G \rightarrow G$, acting by $L_g \tilde{g} = g\tilde{g}$ is a diffeomorphism.

- Def. We say that a vector field X on G is *left-invariant* if $((L_g)_*X) = X$ for any $g \in G$.
- Fact. If X, Y are left-invariant, then also $[X, Y]$ is left-invariant.
- Def. The Lie algebra G' of G is the vector space of left-invariant vector fields on G with algebraic operation given by the commutator.

10. Def. For $X \in G'$ we set $\exp(X) := \gamma_1(e)$.

11. Fact. This exponential map is a diffeomorphism of a neighbourhood of zero in G' into a neighbourhood of e in G .

5.3 From multiplication law in G to algebraic operation in G'

1. Fact. Left-invariance and (79) give $X_g(f) = X_e(f \circ L_g)$ for any $g \in G$. Thus left-invariant vector fields are determined by their values at the neutral element e . In this sense, G' can be identified with T_eG and has the same dimension as G .
2. Let us choose a basis X^1, \dots, X^n in G' . We have

$$[X^A, X^B]_e = \sum_{C=1}^n f^{CAB} X_e^C, \quad (80)$$

where f^{CAB} are called the structure constants. In physics one usually defines the infinitesimal generators⁴ $t^A := iX^A$ and writes (80) as

$$[t^A, t^B] = \sum_{C=1}^n i f^{CAB} t^C, \quad (81)$$

where evaluation at e is understood. We follow the mathematics convention below, unless stated otherwise.

3. Let us determine f^{CAB} from the multiplication law of the group G . We work in some chart $\eta : U \rightarrow O \subset \mathbb{R}^n$ whose domain U is a neighbourhood of e , in which the group elements have the form $g = \eta^{-1}(\varepsilon^1, \dots, \varepsilon^n)$, $e = \eta^{-1}(0, \dots, 0)$. For $\varepsilon_1, \varepsilon_2 \in O$, i.e. $\varepsilon_i = (\varepsilon_i^1, \dots, \varepsilon_i^n)$, $i = 1, 2$, we define the multiplication function

$$m(\varepsilon_1, \varepsilon_2) := \eta(\eta^{-1}(\varepsilon_1)\eta^{-1}(\varepsilon_2)) \quad (82)$$

⁴As we defined G' as a real vector space, it may be unclear what the multiplication by ‘ i ’ means. We recall that it is always possible to ‘complexify’ a real vector space. Furthermore, in the later part of these lectures we will represent Lie algebras on complex vector spaces. Then ‘ i ’ will be provided by the vector space.

which takes values in O i.e. $m(\varepsilon_1, \varepsilon_2) = (m^1(\varepsilon_1, \varepsilon_2), \dots, m^n(\varepsilon_1, \varepsilon_2))$. Since $\eta^{-1}(0) = e$, we have $m(\varepsilon_1, 0) = \varepsilon_1$ and $m(0, \varepsilon_2) = \varepsilon_2$. Consequently

$$\frac{\partial m^i}{\partial \varepsilon_1^j}(0, 0) = \frac{\partial m^i}{\partial \varepsilon_2^j}(0, 0) = \delta_{i,j}. \quad (83)$$

Given a flow of a vector field γ_t in G we define the transported flow $\tilde{\gamma}_t(\varepsilon) = \eta \circ \gamma_t \circ \eta^{-1}(\varepsilon) = (\tilde{\gamma}_t^1(\varepsilon), \dots, \tilde{\gamma}_t^n(\varepsilon))$.

Lemma 5.1 *Let $X^1, \dots, X^A, \dots, X^n$ be vector fields on G whose flows satisfy*

$$\frac{d}{dt} \tilde{\gamma}_{A,t}^i(\varepsilon)|_{t=0} = \frac{\partial m^i}{\partial \varepsilon_2^A}(\varepsilon, 0). \quad (84)$$

Then these fields are left-invariant. Furthermore, they are linearly independent near e and thus span the Lie algebra.

Proof. We want to verify $X_g^A(f) = X_e^A(f \circ L_g)$ for $f \in C^\infty(M)$ and $g = \eta^{-1}(\varepsilon)$. Thus we set $\tilde{f} := f \circ \eta^{-1}$ and compute

$$\begin{aligned} X_{\eta^{-1}(\varepsilon)}^A(f) &= \frac{d}{dt} f(\gamma_{A,t} \circ \eta^{-1}(\varepsilon))|_{t=0} = \frac{d}{dt} \tilde{f}(\tilde{\gamma}_{A,t}(\varepsilon))|_{t=0} \\ &= \sum_{i=1}^n \frac{d}{dt} (\tilde{\gamma}_{A,t}^i(\varepsilon))|_{t=0} \frac{\partial \tilde{f}}{\partial \varepsilon^i}(\varepsilon) = \sum_{i=1}^n \frac{\partial m^i}{\partial \varepsilon_2^A}(\varepsilon, 0) \frac{\partial \tilde{f}}{\partial \varepsilon^i}(\varepsilon). \end{aligned} \quad (85)$$

On the other hand

$$\begin{aligned} X_{\eta^{-1}(0)}^A(f \circ L_{\eta^{-1}(\varepsilon)}) &= \frac{d}{dt} f(L_{\eta^{-1}(\varepsilon)} \gamma_{A,t} \circ \eta^{-1}(0))|_{t=0} \\ &= \frac{d}{dt} f(\eta^{-1}(\varepsilon) \eta^{-1}(\tilde{\gamma}_{A,t}(0)))|_{t=0} = \frac{d}{dt} \tilde{f}(m(\varepsilon, \tilde{\gamma}_{A,t}(0)))|_{t=0} \\ &= \sum_{i,k} \frac{\partial \tilde{f}}{\partial \varepsilon^i}(\varepsilon) \frac{\partial m^i}{\partial \varepsilon_2^k}(\varepsilon, 0) \underbrace{\frac{d}{dt} \tilde{\gamma}_{A,t}^k(0)|_{t=0}}_{\delta_{k,A} \text{ by (83),(84)}}, \end{aligned} \quad (86)$$

which concludes the proof of left-invariance. It suffices to check linear independence near e which follows from (85) and (86). \square

Lemma 5.2 *Under the assumptions of the previous lemma, we have*

$$[X^A, X^B]_e = \sum_{C=1}^n f^{CAB} X_e^C, \quad (87)$$

with $f^{CAB} = \frac{\partial^2 m^C}{\partial \varepsilon_1^A \partial \varepsilon_2^B}(0, 0) - \frac{\partial^2 m^C}{\partial \varepsilon_1^B \partial \varepsilon_2^A}(0, 0)$. (Homework).

5.4 Abstract Lie algebras

1. Def. A Lie algebra is a vector space \mathfrak{g} over the field \mathbb{R} together with a bilinear form $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ which satisfies
 - Antisymmetry: $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{g}$.
 - Jacobi identity: $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$ for all $X, Y, Z \in \mathfrak{g}$.
2. Examples:
 - Let V be a vector space over a field \mathbb{K} . Then $\mathfrak{gl}(V)$ denotes the Lie algebra of all linear mappings $V \rightarrow V$ with $[X, Y] = X \circ Y - Y \circ X$.
 - For $V = \mathbb{K}^n$ we write $\mathfrak{gl}(n, \mathbb{K}) := \mathfrak{gl}(V)$. This is the Lie algebra of all $n \times n$ matrices with entries in \mathbb{K} .
 - $\mathfrak{so}(3) = \{ X \in \mathfrak{gl}(3, \mathbb{R}) \mid X^T = -X \}$ is the Lie algebra of rotations.
 - $\mathfrak{su}(2) = \{ X \in \mathfrak{gl}(2, \mathbb{C}) \mid X^\dagger = -X, \text{Tr}(X) = 0 \}$.
3. Def. Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras. A linear map $h : \mathfrak{g} \rightarrow \mathfrak{h}$ is a Lie algebra homomorphism if $h([X, Y]) = [h(X), h(Y)]$. If h is a bijection, it is called an isomorphism.
4. For example, $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ are isomorphic Lie algebras.
5. Thm. If \mathfrak{g} is a finite-dimensional Lie algebra then there is a unique, up to isomorphism, **simply-connected** Lie group G s.t. $G' = \mathfrak{g}$. (Recall that a topological space is simply connected if any loop can be continuously contracted to a point).
6. Fact: If $H : G \mapsto \hat{G}$ is a Lie group homomorphism then

$$h(X) = \frac{d}{dt} H(\exp(tX))|_{t=0}, \quad X \in G' \tag{88}$$

is a Lie algebra homomorphism. Furthermore $H(\exp(tX)) = \exp(th(X))$ for all $t \in \mathbb{R}$.

7. Thm. Let G, \hat{G} be Lie groups and suppose G is **simply-connected**. Then a Lie algebra homomorphism $h : G' \rightarrow \hat{G}'$ can be lifted to a Lie group homomorphism H .

5.5 Matrix Lie groups

1. Def. A closed subgroup of $GL(n, \mathbb{K})$ is called a matrix Lie group. For example $SO(3)$ and $SU(2)$.
2. Fact: Let G be a matrix Lie group. The Lie algebra G' of this Lie group is given by

$$G' = \{ X \in \mathfrak{gl}(n, \mathbb{K}) \mid e^{tX} \in G \text{ for all } t \in \mathbb{R} \}. \tag{89}$$

We have $GL(n, \mathbb{K})' = \mathfrak{gl}(n, \mathbb{K})$, $SO(3)' = \mathfrak{so}(3)$, $SU(2)' = \mathfrak{su}(2)$. Furthermore, abstract $\exp : G' \rightarrow G$ coincide with the exponential function of a matrix.

3. Recall that $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ are isomorphic Lie algebras. However $SO(3)$ and $SU(2)$ are not isomorphic Lie groups. In particular, $SU(2)$ is simply connected, $SO(3)$ is not.

6 Symmetries II

6.1 Representations

1. Def. A group homomorphism $D : G \rightarrow GL(V)$ is called a representation.
2. Let $D_1 : G \rightarrow GL(V_1)$ and $D_2 : G \rightarrow GL(V_2)$ be two reps of G .

- Def: The direct sum of D_1, D_2 , acting on $V_1 \oplus V_2$ is defined by:

$$(D_1 \oplus D_2)(g)(v_1 \oplus v_2) = (D_1(g)v_1) \oplus (D_2(g)v_2). \quad (90)$$

- Def: The tensor product of D_1, D_2 acting on $V_1 \otimes V_2$ is defined by

$$(D_1 \otimes D_2)(g)(v_1 \otimes v_2) = (D_1(g)v_1) \otimes (D_2(g)v_2). \quad (91)$$

3. The property of irreducibility of a representation D can be explained as follows:
 - Def. We say that a subspace $W \subset V$ is invariant, if $D(g)w \in W$ for any $g \in G$ and $w \in W$,
 - Def. We say that a representation is irreducible if it has no invariant subspaces except for $\{0\}$ and V .
 - Def. A representation is completely reducible if it is a direct sum of irreducible representations.
 - Fact. (Schur Lemma). Let $D : G \rightarrow GL(V)$ be an irreducible representation and V complex. If $A \in GL(V)$ commutes with all $D(g)$, $g \in G$, then it has the form $A = \lambda I$, $\lambda \in \mathbb{C}$.
4. Def: A Lie algebra homomorphism $d : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is called a representation. Irreducibility and complete reducibility of such representations are defined analogously as for groups.
5. Thm. Let $D : G \rightarrow GL(V)$ be a representation of a Lie group. Then

$$d(X) = \frac{d}{dt}D(\exp(tX))|_{t=0}, \text{ for } X \in \mathfrak{g} \quad (92)$$

defines a representation of \mathfrak{g} .

6. Thm. Let G be a **simply-connected** Lie group and \mathfrak{g} its Lie algebra. Let $d : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be a representation. Then there exists a unique representation $D : G \rightarrow GL(V)$ s.t. (92) holds.
7. Def. Let G be a matrix Lie group. We say that a bilinear form $b : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ is invariant, if $b(gXg^{-1}, gYg^{-1}) = b(X, Y)$ for all $g \in G$, $X, Y \in \mathfrak{g}$.

8. Fact. Let $d : G' \rightarrow \mathfrak{gl}(V)$ be a representation, b an invariant bilinear form on G' and $\{X^1, \dots, X^n\}$ a basis in G' . Then the *Casimir operator*

$$C := \sum_{A,B} b(X^A, X^B) d(X^A) d(X^B) \quad (93)$$

is basis-independent and commutes with all $d(X)$, $X \in G'$. Thus, by the Schur Lemma, it is a multiple of unity in any irreducible representation on complex V .

6.2 Projective representations

1. Let V be a complex vector space.

- Def: $U(1) = \{e^{i\varphi} I \mid \varphi \in \mathbb{R}\} \subset GL(V)$.
- Def: $GL(V)/U(1)$ is the (Lie) group generated by the equivalence classes $\underline{A} = \{e^{i\varphi} A \mid \varphi \in \mathbb{R}\}$.
- Def: A homomorphism $\underline{D} : G \rightarrow GL(V)/U(1)$ is called a projective representation (a representation up to a phase).
- For equivalence classes we have $\underline{D}(g_1)\underline{D}(g_2) = \underline{D}(g_1g_2)$. But for a given choice of representatives $D(g) \in \underline{D}(g)$, (s.t. $D(e) = I$, $g \rightarrow D(g)$ continuous)

$$D(g_1)D(g_2) = e^{i\varphi(g_1, g_2)} D(g_1g_2). \quad (94)$$

for some function $\varphi : G \times G \rightarrow \mathbb{R}$.

2. \underline{D} gives rise to the homomorphism $\underline{d} : G' \rightarrow (GL(V)/U(1))'$ given by

$$\underline{d}(X) = \frac{d}{dt} \underline{D}(\exp(tX))|_{t=0}, \quad X \in G'. \quad (95)$$

3. We have $(GL(V)/U(1))' = (GL(V)'/U(1)') = \mathfrak{gl}(V)/(i\mathbb{R})$. The elements of this Lie algebra are the equivalence classes $\underline{Y} = \{Y + iz \mid z \in \mathbb{R}\}$.
4. For equivalence classes we have $[\underline{d}(X^A), \underline{d}(X^B)] = \sum_C f^{CAB} \underline{d}(X^C)$. But for any given choice of representatives $d(X^A) \in \underline{d}(X^A)$

$$[d(X^A), d(X^B)] = \sum_C f^{CAB} d(X^C) - iz^{A,B} I, \quad (96)$$

where $z^{A,B}$ are called the *central charges* and the (-) sign in the last formula is a matter of convention. In the physical notation one defines the infinitesimal generators $T^A = id(X^A)$ so that (96) reads

$$[T^A, T^B] = \sum_C f^{CAB} T^C + iz^{A,B} I. \quad (97)$$

We follow the mathematical convention below unless stated otherwise.

- In some cases it is possible to eliminate $z^{A,B}$ by passing to different representatives $\tilde{d}(X^A) = d(X^A) + ic^A$.
- Then \tilde{d} becomes a Lie algebra representation $G' \rightarrow \mathfrak{gl}(V)$.
- Hence, by a theorem above, \tilde{d} can be lifted to a Lie group representation $\tilde{D} : \tilde{G} \rightarrow GL(V)$, where \tilde{G} is the unique simply-connected Lie group with the Lie algebra G' . (The universal covering group).

5. Let us explain in more detail the concept of the covering space/group:

- Def. A topological space G is path connected, if for any $g_1, g_2 \in G$ there is a continuous map $\gamma : [0, 1] \rightarrow G$ s.t. $\gamma(0) = g_1, \gamma(1) = g_2$.
- Def. A topological space G is simply connected, if it is path connected and every loop in the space can be continuously contracted to a point.
- Def. The universal cover of a connected topological space G is a simply-connected space \tilde{G} together with a covering map $H_c : \tilde{G} \rightarrow G$. The covering map is a local homeomorphism s.t. the cardinal number of $H_c^{-1}(g)$ is independent of g . The universal cover is unique.
- Fact: If G is a Lie group, \tilde{G} is also a Lie group and $H_c : \tilde{G} \rightarrow G$ is a homomorphism s.t. $\ker H_c$ is a discrete subgroup.

6. The situation above occurs in particular for projective unitary representations of $SO(3)$.

- One can choose D s.t. $e^{i\varphi(g_1, g_2)} \in \{\pm 1\}$. By continuity, $e^{i\varphi(g_1, g_2)} = 1$ for g_1, g_2 close to e , hence $z^{A,B} = 0$.
- Thus D can be lifted to a unitary representation \tilde{D} of $\tilde{SO}(3) = SU(2)$. More precisely, $\tilde{D}(A) = D(H_c(A))$, where $A \in SU(2)$ and $H_c : SU(2) \rightarrow SO(3)$ is the covering homomorphism.
- $\ker H_c := H_c^{-1}(e) = \mathbb{Z}_2$ thus $SU(2)/\mathbb{Z}_2 \simeq SO(3)$ and every element of $SO(3)$ corresponds to two elements in $SU(2)$. That is $SU(2)$ is a double covering of $SO(3)$.

6.3 Representations of rotations

1. Fact: $\mathfrak{so}(3) = \{X \in \mathfrak{gl}(3, \mathbb{R}) \mid X^T = -X\}$ is the Lie algebra of $SO(3) = \{R \in GL(3, \mathbb{R}) \mid R^T R = I, \det R = 1\}$. Indeed, let $X \in \mathfrak{so}(3)$. Then

$$(e^{tX})^T e^{tX} = e^{tX^T} e^{tX} = e^{-tX} e^{tX} = 1 \quad (98)$$

$$\det(e^{tX}) = e^{t\text{Tr}X} = 1, \quad (99)$$

where we used that a real anti-symmetric metric has vanishing diagonal elements and consequently is traceless.

2. We choose a basis in $\mathfrak{so}(3)$ as follows

$$L^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad L^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad L^3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (100)$$

so that $e^{\theta \vec{n} \cdot \vec{L}}$ is the rotation around the axis \vec{n} , $\|\vec{n}\| = 1$, by angle θ .

3. One verifies the commutation relations

$$[L^i, L^j] = \varepsilon^{ijk} L^k. \quad (101)$$

These generators are related to the angular momentum operators J^i via $L^i = -iJ^i$. They satisfy accordingly

$$[J^i, J^j] = i\varepsilon^{ijk} J^k. \quad (102)$$

4. Some facts about the irreducible representations of $\mathfrak{so}(3)$:

- From quantum mechanics we know that there is only one Casimir operator $\vec{J}^2 = J_1^2 + J_2^2 + J_3^2$, whose eigenvalues are $j(j+1)$, for $j = 0, 1/2, 1, \dots$
- The irreducible representations $d^{(j)}$ are labelled by j and are $2j + 1$ dimensional.
- The basis vectors are denoted $|j, m\rangle$, $m = -j, -j + 1, \dots, j$, where $d^{(j)}(J_3)|j, m\rangle = m|j, m\rangle$.

5. Recall that $H_c : SU(2) \rightarrow SO(3)$ is the covering homomorphism. It gives rise to the isomorphism $h_c : \mathfrak{su}(2) \rightarrow \mathfrak{so}(3)$ which can be described as follows:

- let $\sigma_1, \sigma_2, \sigma_3$ be the Pauli matrices. Then $Y^j = \frac{1}{2i}\sigma_j$ is a basis of $\mathfrak{su}(2)$, since $[Y^i, Y^j] = \varepsilon^{ijk} Y^k$.
- Then h_c is given in this basis by $h_c(Y^j) = L^j$.
- $h_c^{-1} = d^{(1/2)}$.

6. Since $SU(2)$ is simply-connected, any representation $d^{(j)}$ gives rise to a representation $D^{(j)}$ of $SU(2)$ according to $D^{(j)}(e^{t(h_c)^{-1}(X)}) = e^{td^{(j)}(X)}$. In particular $D^{(1/2)}$, is the defining representation of $SU(2)$ as one can show using that the Pauli matrices are traceless and hermitian.

7. Fact: Only for integer j the representation $D^{(j)}$ of $SU(2)$ can be lifted to a representation of $SO(3)$.

But, as we know from the previous subsection, for any j it can be lifted to a *projective* representation of $SO(3)$. In order to accommodate half-integer spins, we need to justify that the projective representations of $SO(3)$ correspond to rotation symmetry of physical quantum systems. This will be done later today, using that quantum states are determined up to a phase.

8. For any $j = 0, 1/2, 1, \dots$ and a rotation R given by the axis \vec{n} and angle θ we define the *Wigner functions*

$$D_{mm'}^{(j)}(R) := \langle j, m | e^{-i\theta \vec{n} \cdot \vec{J}^{(j)}} | j, m' \rangle, \quad (103)$$

where $\vec{J}^{(j)}$ denotes here the angular momentum in the representation $d^{(j)}$.

9. Let us illustrate how the projective character of $D^{(j)}(R)$ for half-integer j comes about: Consider a rotation $R_{2\pi}$ by 2π around the \vec{e}_3 -axis in the $j = 1/2$ representation. It can be computed in two ways: First, since rotation by 2π is equal to identity, we obtain $D^{(1/2)}(R) = I$. On the other hand, formula (103) gives

$$D^{(1/2)}(R_{2\pi}) = e^{-i2\pi \frac{\sigma_3}{2}} = \exp\left(-i\pi \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = \begin{bmatrix} e^{-i\pi} & 0 \\ 0 & e^{i\pi} \end{bmatrix} = -I. \quad (104)$$

Since we got two different results, $D^{(1/2)}$ is not a well defined homomorphism $SO(3) \rightarrow GL(V)$ (i.e. representation). But it can still be a well defined homomorphism $SO(3) \rightarrow GL(V)/U(1)$ (i.e. projective representation) as the two results differ only by a sign and thus belong to the same equivalence class.

10. Any finite dimensional representation of $SU(2)$ is completely reducible i.e. can be represented as a direct sum of the irreducible representations $D^{(j)}$. In particular, we have

$$D^{(j_1)} \otimes D^{(j_2)} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} D^{(j)}. \quad (105)$$

This is what is called "addition of angular momenta".

6.4 Symmetries of quantum theories

When studying symmetries of a quantum theory, one has to take it seriously that physical states are defined up to a phase. Thus we consider the following setting:

1. \mathcal{H} - Hilbert space of physical states.
2. For $\Psi \in \mathcal{H}$, $\|\Psi\| = 1$ define the ray $\hat{\Psi} := \{ e^{i\theta} \Psi \mid \theta \in \mathbb{R} \}$.
3. $\hat{\mathcal{H}}$ - set of rays with the ray product $[\hat{\Phi} | \hat{\Psi}] := |\langle \Phi | \Psi \rangle|^2$.

Definition 6.1 A *symmetry transformation* of a quantum system is an invertible map $\hat{U} : \hat{\mathcal{H}} \rightarrow \hat{\mathcal{H}}$ s.t. $[\hat{U}\hat{\Phi} | \hat{U}\hat{\Psi}] = [\hat{\Phi} | \hat{\Psi}]$. Such transformations form a group.

Theorem 6.2 (Wigner) For any symmetry transformation $\hat{U} : \hat{\mathcal{H}} \rightarrow \hat{\mathcal{H}}$ we can find a unitary or anti-unitary operator $U : \mathcal{H} \rightarrow \mathcal{H}$ s.t. $\hat{U}\hat{\Psi} = \widehat{U\Psi}$. U is unique up to phase.

Anti-unitary operators are defined as follows:

- A map $U : \mathcal{H} \rightarrow \mathcal{H}$ is anti-linear if $U(c_1\Psi_1 + c_2\Psi_2) = \bar{c}_1U\Psi_1 + \bar{c}_2U\Psi_2$.
- The adjoint of an anti-linear map is given by $\langle \Phi|U^\dagger\Psi \rangle = \overline{\langle U\Phi|\Psi \rangle}$.
- An anti-linear map is called anti-unitary if $U^\dagger U = UU^\dagger = I$ in which case $\langle U\Phi|U\Psi \rangle = \overline{\langle \Phi|U^\dagger U\Psi \rangle} = \langle \Psi|\Phi \rangle$.

Anti-unitary operators will be needed to implement discrete symmetries, e.g. time reversal. For symmetries described by connected Lie groups anti-unitary operators can be excluded, as we indicate below.

Application of the Wigner theorem:

1. Suppose a connected Lie group G is a symmetry of our theory i.e. there is a group homomorphism $G \ni g \mapsto \hat{U}(g)$ into symmetry transformations.
2. The Wigner theorem gives corresponding unitary operators $U(g)$. Since they are determined up to a phase, they form only a *projective* representation:

$$U(g_1)U(g_2) = e^{i\theta_{1,2}}U(g_1g_2). \quad (106)$$

(Since G is connected, we can exclude that some $U(g)$ are anti-unitary. Indeed for a connected group we have $g = g_0^2$ for some $g_0 \in G$. Now $U(g) = e^{-i\theta}U(g_0)U(g_0)$ which is unitary no matter if $U(g_0)$ is unitary or anti-unitary).

3. As discussed above⁵, for a large class of connected Lie groups G (including $SO(3)$ and \mathcal{P}_+^\uparrow) a projective unitary representation of G corresponds to an *ordinary* unitary representation of the covering group \tilde{G}

$$\tilde{G} \ni \tilde{g} \mapsto \tilde{U}(\tilde{g}) \in B(\mathcal{H}). \quad (107)$$

In particular, projective unitary representations of $SO(3)$ correspond to ordinary unitary representations of $SU(2)$ and thus there is room for half-integer spin!

⁵Strictly speaking, in previous sections we tacitly assumed that the representations act on finite-dimensional vector spaces, while here \mathcal{H} can be infinite dimensional. Fortunately, the relevant results generalize.

7 Mathematical framework for QED

Main reference for this section is [17, Chapter 7].

7.1 Classical field theory

Consider a Lagrangian $\mathcal{L}(\phi, \phi^*, \partial\phi, \partial\phi^*; A, \partial A)$ where A is a vector field, ϕ a complex scalar field. Suppose first that \mathcal{L} is invariant under the transformations

$$\phi_\varepsilon(x) = e^{i\varepsilon(x)}\phi(x), \quad \phi_\varepsilon^*(x) = e^{-i\varepsilon(x)}\phi^*(x), \quad A_{\mu,\varepsilon}(x) = A_\mu(x) + \partial_\mu\varepsilon(x), \quad (108)$$

for some function ε . Thus the corresponding variation of the Lagrangian must vanish. Exploiting the Euler-Lagrange equations we get

$$\delta\mathcal{L} = (\partial_\mu j^\mu)\varepsilon + (j^\nu - \partial_\mu F^{\mu\nu})\partial_\nu\varepsilon - (F^{\mu\nu})\partial_\mu\partial_\nu\varepsilon, \quad (109)$$

where

$$j^\mu(x) := \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(x)i\phi(x) - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}(x)i\phi^*(x), \quad F^{\mu\nu} := -\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} \quad (110)$$

- From the $\partial_\mu\partial_\nu\varepsilon$ -term of the (109) we obtain that the symmetric part of $F^{\mu\nu}$ vanishes, i.e.

$$F^{\mu\nu} = -F^{\nu\mu} \quad (111)$$

- From the $\partial_\nu\varepsilon$ -term of the (109) we get the *local Gauss Law*

$$\partial_\mu F^{\mu\nu} = j^\nu \quad (112)$$

- From the ε -term of (109) we get $\partial_\mu j^\mu = 0$ and $\partial_i Q = 0$, where

$$Q = \int d^3y j^0(0, \vec{y}). \quad (113)$$

(Noether's theorem)

- Furthermore, Q is the infinitesimal generator of the global $U(1)$ symmetry, i.e.

$$\{Q, \phi(0, \vec{x})\} = -\frac{d}{d\varepsilon}\phi_\varepsilon(0, \vec{x})|_{\varepsilon=0} = -i\phi(0, \vec{x}), \quad (114)$$

$$\{Q, \phi^*(0, \vec{x})\} = -\frac{d}{d\varepsilon}\phi_\varepsilon^*(0, \vec{x})|_{\varepsilon=0} = i\phi^*(0, \vec{x}). \quad (115)$$

Here the Poisson bracket is defined by

$$\{F, G\} = \int d^3z \left(\frac{\delta F}{\delta\phi(0, \vec{z})} \frac{\delta G}{\delta\pi(0, \vec{z})} - \frac{\delta F}{\delta\pi(0, \vec{z})} \frac{\delta G}{\delta\phi(0, \vec{z})} \right) + \dots \quad (116)$$

where $\pi(z) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)(z)}$ is the canonical momentum and the omitted terms correspond to ϕ^* and A . (Note, however, that terms corresponding to A are not relevant for (115)).

- On the other hand using $\partial_\mu F^{\mu\nu} = j^\nu$ we can compute

$$\begin{aligned}
-i\phi(0, \vec{x}) &= \{Q, \phi(0, \vec{x})\} = \int d^3y \{j^0(0, \vec{y}), \phi(0, \vec{x})\} \\
&= \int d^3y \{\partial_i F^{i,0}(0, \vec{y}), \phi(0, \vec{x})\} = \lim_{R \rightarrow \infty} \int_{B_R} d^3y \{\vec{\nabla} \cdot \vec{E}(0, \vec{y}), \phi(0, \vec{x})\} \\
&= \lim_{R \rightarrow \infty} \int_{\partial B_R} d\vec{\sigma}(\vec{y}) \cdot \{\vec{E}(0, \vec{y}), \phi(0, \vec{x})\}, \tag{117}
\end{aligned}$$

where $\vec{E} := (F^{1,0}, F^{2,0}, F^{3,0})$, B_R is a ball of radius R centered at zero, ∂B_R its boundary (a sphere) and we used the Stokes theorem. Note that in quantum theory, where $\{\cdot, \cdot\} \rightarrow -i[\cdot, \cdot]$ above⁶, the last expression would be zero by locality, giving a contradiction!

- One possible way out (which we will not follow) is to abandon locality of charged fields but keep $\partial_\mu F^{\mu\nu} = j^\nu$ (Quantisation in the Coulomb gauge).
- We will follow instead the Gupta-Bleuler approach, where all fields are local, but $\langle \Psi_1 | (\partial_\mu F^{\mu\nu} - j^\nu) \Psi_2 \rangle = 0$ only for Ψ_1, Ψ_2 in some ‘physical subspace’ $\mathcal{H}' \subset \mathcal{H}$. This will enforce $\langle \Psi | \Psi \rangle < 0$ for some $\Psi \in \mathcal{H}$ thus we have to use ‘indefinite metric Hilbert spaces’ (Krein spaces)
- Incidentally, local, Poincaré covariant massless vector fields A^μ do exist on Krein spaces (which is not the case on Hilbert spaces). Thus we will have candidates for the electromagnetic potential.

7.2 Strocchi-Wightman framework [18, 19]

Definition 7.1 An ‘indefinite metric Hilbert space’ (Krein space) \mathcal{H} is a vector space equipped with a sesquilinear form $\langle \cdot | \cdot \rangle$ s.t.

- $\langle \cdot | \cdot \rangle$ is non-degenerate, i.e. for any $\Psi \neq 0$ there is some $\Phi \in \mathcal{H}$ s.t. $\langle \Psi | \Phi \rangle \neq 0$.
- \mathcal{H} carries an auxiliary positive-definite scalar product $(\cdot | \cdot)$ w.r.t. which it is a Hilbert space.
- There is a bounded, invertible operator η on \mathcal{H} , self-adjoint w.r.t. $(\cdot | \cdot)$, s.t. $\langle \Psi_1 | \Psi_2 \rangle = (\Psi_1 | \eta \Psi_2)$.

Only the first property above is physically important. The role of the last two properties is to provide a topology on \mathcal{H} which is needed for technical reasons (e.g. density of various domains).

Definition 7.2 A Strocchi-Wightman relativistic quantum mechanics is given by:

⁶It should be mentioned that for gauge theories the standard quantisation prescription $\{\cdot, \cdot\} \rightarrow -i[\cdot, \cdot]$ may fail in general and a detour via ‘Dirac brackets’ is required. However, in the above situation the simple analogy can be maintained.

1. A Krein space \mathcal{H} .
2. A physical subspace $\mathcal{H}' \subset \mathcal{H}$ s.t. $\langle \Psi | \Psi \rangle \geq 0$ for $\Psi \in \mathcal{H}'$.
3. The physical Hilbert space $\mathcal{H}_{\text{ph}} := (\mathcal{H}'/\mathcal{H}'')^{\text{cp1}}$, where $\mathcal{H}'' := \{\Psi \in \mathcal{H}' \mid \langle \Psi | \Psi \rangle = 0\}$. Its elements are equivalence classes $[\Psi] = \{\Psi + \Psi_0 \mid \Psi_0 \in \mathcal{H}''\}$, where $\Psi \in \mathcal{H}'$.
4. A $\langle \cdot | \cdot \rangle$ -unitary representation $\tilde{\mathcal{P}}_+^\dagger \ni (\tilde{\Lambda}, a) \mapsto U(\tilde{\Lambda}, a)$ in \mathcal{H} s.t. \mathcal{H}' is invariant under U . Then U induces a unitary representation on \mathcal{H}_{ph} by $U_{\text{ph}}(\tilde{\Lambda}, a)[\Psi] = [U(\tilde{\Lambda}, a)\Psi]$. We assume that U_{ph} is continuous and satisfies the spectrum condition.
5. A unique (up to phase) vacuum vector $\Omega \in \mathcal{H}'$ s.t. $\langle \Omega | \Omega \rangle = 1$ and $U(\tilde{\Lambda}, a)\Omega = \Omega$.

Definition 7.3 A Strocchi-Wightman QFT is given by:

1. A Strocchi-Wightman relativistic QM $(\mathcal{H}, \mathcal{H}', U, \Omega)$.
2. A family of operator-valued distributions $(\phi_\ell^{(\kappa)}, D)$, $\kappa \in \mathbb{I}$, $\ell = 1, 2, \dots, r_\kappa$ s.t.
 - \mathbb{I} is some finite or infinite collection of indices numbering the types of the fields corresponding to finite-dimensional representations $D^{(\kappa)}$ of $\tilde{\mathcal{L}}_+^\dagger = SL(2, \mathbb{C})$.
 - For a fixed $\kappa \in \mathbb{I}$ the field $\phi^{(\kappa)} = (\phi_\ell^{(\kappa)})_{\ell=1, \dots, r_\kappa}$ transform under $D^{(\kappa)}$.
 - For any κ, ℓ there exists some $\bar{\kappa}, \bar{\ell}$ s.t. $\phi_\ell^{(\kappa)}(f)^\dagger = \phi_{\bar{\ell}}^{(\bar{\kappa})}(\bar{f})$.
 - $\Omega \in D$ and $U(\tilde{\Lambda}, a)D \subset D$ for all $(\tilde{\Lambda}, a) \in \tilde{\mathcal{P}}_+^\dagger$.

satisfying:

(a) (Locality) If $\text{supp } f_1$ and $\text{supp } f_2$ are spacelike separated, then

$$[\phi_\ell^{(\kappa)}(f_1), \phi_{\ell'}^{(\kappa')}(\bar{f}_2)]_- = 0 \text{ or } [\phi_\ell^{(\kappa)}(f_1), \phi_{\ell'}^{(\kappa')}(\bar{f}_2)]_+ = 0 \quad (118)$$

in the sense of weak commutativity on D . (Here $-/+$ refers to commutator/anti-commutator).

(b) (Covariance) For all $(\tilde{\Lambda}, a) \in \tilde{\mathcal{P}}_+^\dagger$ and $f \in S$

$$U(\tilde{\Lambda}, a)\phi_\ell^{(\kappa)}(f)U(\tilde{\Lambda}, a)^\dagger = \sum_{\ell'} D_{\ell\ell'}^{(\kappa)}(\tilde{\Lambda}^{-1})\phi_{\ell'}^{(\kappa)}(f_{(\Lambda, a)}). \quad (119)$$

Here $f_{(\Lambda, a)}(x) = f(\Lambda^{-1}(x - a))$.

(c) (Cyclicity of the vacuum) $\mathcal{D} = \text{Span}\{\phi_{\ell_1}^{(\kappa_1)}(f_1) \dots \phi_{\ell_m}^{(\kappa_m)}(f_m)\Omega \mid f_1, \dots, f_m \in S, m \in \mathbb{N}_0\}$ is a dense subspace of \mathcal{H} in the topology given by $(\cdot | \cdot)$.

The distributions $(\phi_\ell^{(\kappa)}, D)$ are called the Strocchi-Wightman quantum fields.

7.3 Free field examples

7.3.1 Free Wightman fields

1. Suppose first that \mathcal{H} is a Hilbert space (i.e. $\langle \cdot | \cdot \rangle$ is a positive-definite scalar product and we can choose $\mathcal{H}' = \mathcal{H}$). Then the above setting is called the Wightman framework for fields with arbitrary (finite) spin.
2. Let us stay for a moment in this Hilbert space framework. Recall that finite-dimensional irreducible representations $D^{(\kappa)}$ of $\tilde{\mathcal{L}}_+^\dagger$ are labelled by two numbers (A, B) . From the physics lecture you know the following free field examples:
 - $(0, 0)$: scalar field ϕ
 - $(\frac{1}{2}, \frac{1}{2})$: massive vector field j^μ
 - $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$: Dirac field ψ
 - $(1, 0) \oplus (0, 1)$: Faraday tensor $F^{\mu\nu}$.
3. It is however not possible to construct a massless free vector field A^μ on a Hilbert space which is local and Poincaré covariant [25]. Such fields turn out to exist on Krein spaces, which is usually given as the main reason to introduce them.

7.3.2 Free massless vector field A^μ on Krein space

The Gupta-Bleuler electromagnetic potential has the form

$$A_\mu(x) = \int \frac{d^3k}{2k_0(2\pi)^3} \sum_{\lambda=0}^3 [a^{(\lambda)}(k) \varepsilon_\mu^{(\lambda)}(k) e^{-ikx} + a^{(\lambda)\dagger}(k) \varepsilon_\mu^{(\lambda)*}(k) e^{ikx}], \quad (120)$$

where $k_0 = |k|$ and $\varepsilon_\mu^{(\lambda)}$ are polarisation vectors which satisfy the orthogonality and completeness relations

$$\varepsilon^{(\lambda),\mu}(k) \cdot \varepsilon_\mu^{(\lambda')*}(k) = g^{\lambda\lambda'}, \quad \sum_{\lambda} (g^{\lambda\lambda})^{-1} \varepsilon_\mu^{(\lambda)}(k) \cdot \varepsilon_\nu^{(\lambda)*}(k) = g_{\mu\nu}. \quad (121)$$

For the $a^\lambda, a^{(\lambda)\dagger}$ we have

$$[a^{(\lambda)}(k), a^{(\lambda')\dagger}(k')] = -g^{\lambda\lambda'} 2k^0 (2\pi)^3 \delta(\vec{k} - \vec{k}'). \quad (122)$$

Due to $-g^{00} = -1$ we have $\langle a^{(0)\dagger}(f) \Omega | a^{(0)\dagger}(f) \Omega \rangle < 0$ thus our ‘Fock space’ \mathcal{H} turns out to be a Krein space. Furthermore, the ‘photons’ above have four polarisations and not two. These unphysical degrees of freedom will be eliminated by the Gupta-Bleuler subsidiary condition (127) below.

Let us point out another peculiarity of this potential: We can form $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ so that $\varepsilon^{\alpha\beta\mu\nu} \partial_\beta F_{\mu\nu}(x) = 0$ is automatic. But the remaining free Maxwell equations fail:

$$\partial_\mu F^{\mu\nu}(x) = -\partial^\nu (\partial_\rho A^\rho)(x) \neq 0. \quad (123)$$

This can be expected on general grounds:

Theorem 7.4 (Strocchi [20, 21]) Any Strocchi-Wightman vector field A_μ with $\partial_\mu F^{\mu\nu}(x) = 0$ is trivial, i.e. $\langle \Omega | F^{\mu\nu}(x) F^{\alpha\beta}(y) \Omega \rangle = 0$.

This highlights the necessity of a gauge-fixing term in the Lagrangian, which is another point to which we will come below in the context of interacting QED.

7.4 Quantum Electrodynamics

Definition 7.5 QED is a Strocchi-Wightman QFT whose fields include $F^{\mu\nu}, j^\mu$ and some ‘charged fields’ $\phi^{(\kappa)}$ s.t. the physical subspace \mathcal{H}' satisfies:

(i) There is a dense domain $D' \subset \mathcal{H}'$ s.t. $F^{\mu\nu}(f)D' \subset D', j^\mu(f)D' \subset D'$ and $U(a, \tilde{\Lambda})D' \subset D'$.

(ii) For $\Psi_1 \in \mathcal{H}'$ and $\Psi_2 \in D'$

$$\langle \Psi_1 | (\partial_\mu F^{\mu\nu} - j^\nu)(f) \Psi_2 \rangle = 0, \quad \langle \Psi_1 | (\varepsilon_{\mu\nu\rho\sigma} \partial^\nu F^{\rho\sigma})(f) \Psi_2 \rangle = 0. \quad (124)$$

For QED defined as above, one can define the electric charge operator by suitably regularizing $Q = \int d^3y j^0(0, \vec{y})$.

Theorem 7.6 (Strocchi-Picasso-Ferrari [22]) Suppose that Q is an infinitesimal generator of the global $U(1)$ symmetry, i.e. for some field ϕ

$$\phi(x) = [Q, \phi(x)] \text{ on } \mathcal{H} \quad (125)$$

and that $\langle \Psi_1 | \phi(x) \Psi_2 \rangle \neq 0$ for some $\Psi_1, \Psi_2 \in \mathcal{H}'$. Then

1. There is $\Psi \in D'$ s.t. $(\partial_\mu F^{\mu\nu} - j^\nu)(x)\Psi \neq 0$.
2. There is $0 \neq \Psi \in \mathcal{H}'$ s.t. $\langle \Psi | \Psi \rangle = 0$, i.e. $\mathcal{H}'' \neq \{0\}$.
3. There is $\Psi \in \mathcal{H}$ s.t. $\langle \Psi | \Psi \rangle < 0$.

The proof is simple and relies on a Stokes theorem computation analogous to (117). This theorem shows that the Maxwell equations can hold on \mathcal{H}' at best in matrix elements and that the Krein space framework is needed in (local) QED also in the presence of interactions.

Definition 7.7 We say that QED is in the Gupta-Bleuler gauge if it contains (in addition to other fields) a vector field A_μ s.t. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and

$$\partial_\mu F^{\mu\nu} - j^\nu = -\partial^\nu(\partial_\rho A^\rho) \quad (126)$$

holds as an operator identity on \mathcal{H} . Furthermore, the physical subspace is chosen as

$$\mathcal{H}' := \{ \Psi \in \mathcal{H} \mid (\partial_\rho A^\rho)^{(+)}(f)\Psi = 0 \text{ for all } f \in S \}, \quad (127)$$

where $(\partial_\rho A^\rho)^{(+)}$ is the positive frequency part of $(\partial_\rho A^\rho)$.

We add several remarks on this definition:

1. Note that by applying ∂_ν to (126) and using current conservation we obtain $\square(\partial_\rho A^\rho) = 0$, thus the decomposition into positive and negative frequency parts is meaningful. For this reason, (124) formally hold. But positivity of the scalar product on \mathcal{H}' needs to be assumed. (Known only for the free electromagnetic field).
2. The equation (126) comes from a classical Lagrangian with the gauge-fixing term, e.g.

$$\mathcal{L}_{\text{gf}} = (D^\mu \phi)^*(D_\mu \phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2. \quad (128)$$

\mathcal{L}_{gf} is still invariant under ‘residual’ local gauge transformations s.t. $\square\varepsilon(x) = 0$. Denote infinitesimal transformations of the fields as $\delta_\varepsilon\phi(x) = i\varepsilon(x)\phi(x)$, $\delta_\varepsilon\phi(x)^* = -i\varepsilon(x)\phi(x)^*$ and $\delta_\varepsilon A_\mu(x) = \partial_\mu\varepsilon(x)$.

3. Def. Let \mathcal{A} denote the algebra spanned by polynomials of quantum fields A^μ , j^μ , ϕ , ϕ^* smeared with smooth, compactly supported functions. We extend δ_ε to \mathcal{A} via the Leibniz rule and denote the subalgebra of (residual) gauge-invariant elements by \mathcal{A}_{gi} .
4. It turns out, that a ‘local vector’ (i.e. $\Psi = A\Omega$, where $A \in \mathcal{A}$) belongs to \mathcal{H}' iff $A \in \mathcal{A}_{\text{gi}}$ [17].

7.5 Electrically charged states of QED

Important problem in QED is a construction of physical electrically charged states. Vectors of the form $\phi(f)\Omega$, where ϕ is a charged field, are not in \mathcal{H}' , because ϕ is not invariant under residual gauge transformations. Moreover:

Proposition 7.8 [17] *For any local vectors $\Psi, \Phi \in \mathcal{H}'$ in QED we have $\langle \Psi | Q\Phi \rangle = 0$.*

We face the problem of constructing a field ϕ_C which is invariant under (residual) local gauge transformations (so that $\phi_C(f)\Omega$ is ‘close’ to \mathcal{H}') and non-invariant under global gauge transformations (so that $\phi_C(f)\Omega$ is charged). Here is a candidate:

$$\phi_C(x) := e^{i[(\Delta)^{-1}\partial_i A^i](x)}\phi(x) = e^{i[(\Delta)^{-1}\partial_i(A^i + \partial^i \varepsilon)](x)}e^{i\varepsilon(x)}\phi(x), \quad (129)$$

where the last equality holds for local gauge transformations but fails for global (i.e. $\varepsilon(x)=\text{const}$). ϕ_C is simply the (non-local) charged field in the Coulomb gauge expressed in terms of the Gupta-Bleuler fields. Indeed, we can see (129) as a gauge transformation. Then the corresponding transformation applied to the potential gives

$$A_{\mu,C}(x) = A_\mu(x) + \partial_\mu[(\Delta)^{-1}\partial_i A^i](x). \quad (130)$$

which satisfies $\vec{\nabla} \cdot \vec{A}_C = 0$. Then $\phi_C(f)\Omega$ is a candidate for an electrically charged states. Since $\phi_C(f)$ is a very singular objects, this vector ‘escapes’ from \mathcal{H} and its control (in the perturbative or axiomatic setting) requires subtle mathematical methods (cf. [17, 18]). These are outside of the scope of these lectures.

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