A crash course in functional analysis

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May 3, 2017

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1 Measure theory

The theory of self-adjoint operators relies heavily on measure theory. Here we recall several basic concepts and facts which will be useful in these lectures. Proofs can be found in the first two chapters of [3].

1. Let X be a topological space (a set with topology). A family $\mathcal{M}$ of subsets of X is a σ-algebra in X if it has the following properties:

   - $X \in \mathcal{M}$,
   - $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$,
   - $A_n \in \mathcal{M}$, $n \in \mathbb{N}$, $\Rightarrow A := \bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$.

   If $\mathcal{M}$ is a σ-algebra in X then X is called a *measurable space* and elements of $\mathcal{M}$ are called *measurable sets.*

   Remark 1.1. *The Borel σ-algebra is the smallest σ-algebra containing all open sets of X. Its elements are called Borel sets.*

2. Let X be a measure space and Y a topological space. Then a map $f : X \to Y$ is called *measurable* if for any open $V \subset Y$ the inverse image $f^{-1}(V)$ is a measurable set.

3. A (positive) measure is a function $\mu : \mathcal{M} \to [0, \infty]$ s.t. for any countable family of disjoint sets $A_i \in \mathcal{M}$ we have

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i). \tag{1.1}$$

Also, we assume that $\mu(A) < \infty$ for at least one $A \in \mathcal{M}$. Moreover, we say that a *measure space* is a measurable space whose σ-algebra of measurable sets carries a positive measure.

4. We denote by $\mathcal{L}^p(X, \mu)$, $1 \leq p < \infty$ the space of measurable functions $f : X \to \mathbb{C}$ s.t.

$$\|f\|_p := \left(\int_X |f(x)|^p d\mu(x) \right)^{1/p} < \infty. \tag{1.2}$$

We denote by $L^p(X, \mu)$ the space of equivalence classes of functions from $\mathcal{L}^p(X, \mu)$ which are equal almost everywhere w.r.t. μ. Space $L^p(X, \mu)$ is a Banach space with the norm (1.2) (Riesz-Fisher theorem).

Theorem 1.2. *(Riesz-Markov-Kakutani). Let X be a locally compact Hausdorff space*[1] *and $C_c(X)$ the space of continuous compactly supported functions on X. Let $\Lambda : C_c(X) \to \mathbb{C}$ be a positive linear functional*[2]*. Then there exists a σ-algebra $\mathcal{M}$ in X and a positive measure on $\mathcal{M}$ s.t.*

$$\Lambda(f) = \int_X f(x) d\mu(x) \text{ for any } f \in C_c(X). \tag{1.3}$$

Theorem 1.3. *(Dominated convergence). Let f_n be a sequence of complex, measurable functions on X s.t.*

$$f(x) = \lim_{n\to\infty} f_n(x) \tag{1.4}$$

exists for any x. If there exists a function $g \in \mathcal{L}^1(X, \mu)$ s.t.

$$|f_n(x)| \leq g(x) \text{ for all } n \in \mathbb{N}, x \in X, \tag{1.5}$$

then $f \in \mathcal{L}^1(X, \mu)$. Moreover,

$$\lim_{n\to\infty} \int_X f_n(x) d\mu(x) = \int_X f(x) d\mu(x). \tag{1.6}$$

2 Spectral theorem

Definition 2.1. *Let X be a measurable space with a σ-algebra $\mathcal{M}$. We say that $\mathcal{M} \ni \Delta \to E(\Delta) \in B(\mathcal{H})$ is a spectral measure if:*

- *Each $E(\Delta)$ is an orthogonal projection.*
- $E(\emptyset) = 0$, $E(X) = 1$.

[1] i.e. a topological space s.t. any two distinct points have disjoint neighbourhoods and any point has a compact neighbourhood.

[2] i.e. if f takes values in $[0, \infty]$ then $\Lambda(f) \geq 0$.

- *If* $\Delta = \bigcup_{n=1}^{N} \Delta_n$*, with* $\Delta_n \cap \Delta_m = \emptyset$ *for* $n \neq m$*, then*

$$E(\Delta) = \sum_{n=1}^{N} E(\Delta_n). \tag{2.7}$$

  For $N = \infty$ *this reads* $\mathrm{s-}\lim_{N\to\infty} \sum_{n=1}^{N} E(\Delta_n)$ *, where strong limit means limit on any fixed vector.*

- $E(\Delta_1)E(\Delta_2) = E(\Delta_1 \cap \Delta_2)$.

For any $\psi \in \mathcal{H}$ the expression $\Delta \to \langle \psi, E(\Delta)\psi \rangle$ is a positive measure and the formula

$$\langle \psi, A\psi \rangle = \int x \, \langle \psi, dE(x)\psi \rangle \tag{2.8}$$

defines a self-adjoint operator A on the domain

$$D(A) = \{\psi \in \mathcal{H} \,|\, \int |x|^2 \langle \psi, dE(x)\psi \rangle < \infty \}. \tag{2.9}$$

It turns out that also the converse is true:

Theorem 2.2. *(Spectral theorem, spectral measure variant). For any self-adjoint operator* $(A, D(A))$ *there exists a spectral measure* E *on a* σ*-algebra of (Borel-)measurable sets on* $\mathbb{R}$ *s.t.*

$$A = \int x dE(x), \tag{2.10}$$

where the last relation means that (2.8), (2.9) hold. Furthermore $\mathrm{Sp}(A) = \mathrm{supp}\, dE$ *i.e. the spectrum of* A *equals the support of the spectral measure.*

Idea of proof: Let A be bounded, for simplicity. Consider a map $g \to \langle \psi, g(A)\psi \rangle$ defined first for polynomials and then extended, using the Stone-Weierstrass theorem to continuous functions. Then we get by the Riesz-Markov-Kakutani theorem a measure space (X_ψ, μ_ψ) s.t.

$$\langle \psi, g(A)\psi \rangle = \int_{X_\psi} g(x) d\mu_\psi(x) \tag{2.11}$$

and we can extend this expression to measurable functions g. In particular, we set $E(\Delta) := \chi_\Delta(A)$ and it is easy to check that this gives a spectral measure. $\square$

In this outline of a proof we also learned how to form functions of operators

$$g(A) = \int g(x) dE(x) \tag{2.12}$$

Example: $A = \mathrm{diag}(\lambda_1, \lambda_2)$. For the corresponding eigenvectors $|e_1\rangle = (1, 0)$ and $|e_2\rangle = (0, 1)$ we have

$$A = \lambda_1 |e_1\rangle\langle e_1| + \lambda_2 |e_2\rangle\langle e_2| = \int \lambda dE(\lambda) \tag{2.13}$$

with $dE(\lambda) = (|e_1\rangle\langle e_1|\delta(\lambda - \lambda_1) + |e_2\rangle\langle e_2|\delta(\lambda - \lambda_1))d\lambda$. Then clearly $g(A) = \mathrm{diag}(g(\lambda_1), g(\lambda_2))$ i.e.

$$g(A) = g(\lambda_1)|e_1\rangle\langle e_1| + g(\lambda_2)|e_2\rangle\langle e_2| = \int g(\lambda) dE(\lambda) \tag{2.14}$$

Fact. If s.a. $A \notin \mathbb{C}I$ then there is a Borel set Δ s.t. $0 \neq \chi_\Delta(A) \neq I$. (This was used in the proof of equivalence of two notions of irreducibility).
Proof. Recall that $\chi_\Delta(A) =: E(\Delta)$. Take $\lambda \in \mathrm{Sp}(A)$. $\mathrm{Sp}(A)$ equals the support of dE. This means that for any ball $B(\lambda, r)$ of radius $r > 0$ centered at λ we have $E(B(\lambda, r)) \neq 0$. Suppose $E(B(\lambda, r)) = I$ for all $r > 0$. Then $I = E(\mathbb{R}) = E(B(\lambda, r) \cup (\mathbb{R} \backslash B(\lambda, r))) = I + E(\mathbb{R} \backslash B(\lambda, r)))$ by definition of the spectral measure . That is $E(\mathbb{R} \backslash B(\lambda, r)) = 0$ so $\mathrm{Sp}(A) = \{\lambda\}$ i.e. $A = \lambda I$. $\square$

3 Stone theorem

Definition 3.1. $\mathbb{R} \ni t \mapsto U(t) \in B(\mathcal{H})$ *is called a (one parameter) strongly continuous group of unitaries if all the* $U(t)$ *are unitary operators and*

1. $U(t_1 + t_2) = U(t_1)U(t_2)$ *for all* $t_1, t_2 \in \mathbb{R}$.
2. $U(0) = I$.
3. $t \to U(t)\psi$ *is continuous for any* $\psi \in \mathcal{H}$ *(i.e. strongly continuous).*

Example: Given a s.a. operator A, one can use the spectral theorem to define

$$U(t) = e^{itA} \tag{3.15}$$

This is a strongly continuous group of unitaries. As for continuity, the key observation is that for $t_n \to 0$

$$\lim_{n\to\infty} \langle \psi, (U(t_n) - I)\psi \rangle = \lim_{n\to\infty} \int (e^{it_n\lambda} - 1)\langle \psi, dE(\lambda)\psi \rangle = 0 \tag{3.16}$$

by the dominated convergence theorem. It turns out that this example characterizes all strongly continuous groups of unitaries:

Theorem 3.2. *(Stone) Given a strongly continuous group of unitaries* U *there exists a self-adjoint operator* A *s.t.*

$$U(t) = e^{itA}. \tag{3.17}$$

Furthermore $D(A) = \{\psi \in \mathcal{H} \mid \lim_{t\to 0} \frac{(U(t)-I)}{t}\psi$ *exists.* $\}$.

References

[1] M. Reed, B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis*. Academic Press, 1980.

[2] M. Reed, B. Simon, *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness*. Academic Press 1975.

[3] W. Rudin, *Real and Complex Analysis*. McGraw-Hill Book Company, 1987.