Generalized Wentzell boundary conditions and holography

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Introduction

We study a scalar field ϕ subject to the action

$$\mathcal{S} = -\frac{1}{2}\int_M \left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \mu^2\phi^2\right)\mathrm{d}^{d+1}x - \frac{c}{2}\int_{\partial M}\left(h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \mu^2\phi^2\right)\mathrm{d}^d x$$

Specifically, $M = \mathbb{R}\times\Sigma$ with $\Sigma = \mathbb{R}^{d-1}\times[-S,S]$ (but also general $\Sigma\subset\mathbb{R}^d$).

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[c] = Length, c > 0 !!

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- The Nambu-Goto string with masses at the ends [Chodos & Thorn 74]:

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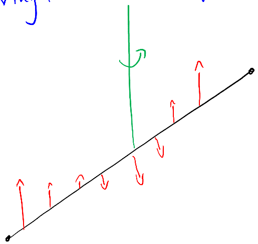
String tension

mass at bdy

Fluctuations
⊥ to plane of rotation
have action
[2.13]



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- Counterterms in the AdS/CFT correspondence [Balasubramanian & Kraus 99]:

$$\mathcal{S} = -\frac{1}{16\pi G}\int_M \sqrt{g}\left(R_g - \tfrac{12}{\ell^2}\right)\mathrm{d}^5x - \frac{1}{8\pi G}\int_{\partial M}\sqrt{h}\left(\Theta - \tfrac{\ell}{4}R_h + \tfrac{3}{\ell}\right)\mathrm{d}^4x.$$

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Note sign!
EH with c.c.
extr. curv. GHY term
EH with c.c

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- Holographic renormalization [Skenderis et al]

$$\mathcal{S} = -\frac{1}{2}\int_{\rho\geqslant\varepsilon}\sqrt{g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \left(\tfrac{d^2}{4}-1\right)\phi^2\right)\mathrm{d}^{d+1}x$$
$$-\frac{\ell}{2}\int_{\partial M_\varepsilon}\sqrt{h}\left(\tfrac{1}{2}\log\varepsilon\, h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \left(\tfrac{d}{2}-1\right)\phi^2\right)\mathrm{d}^d x.$$

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< 0 !!

Questions

- Is the classical system well-behaved, i.e., is the Cauchy problem well-posed?
- Can one quantize the system? If yes, what is the interplay between bulk and boundary fields?

Outline

The wave equation

Quantization

Conclusion

Variation of

$$\mathcal{S} = -\frac{1}{2}\int_M \left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \mu^2\phi^2\right)\mathrm{d}^{d+1}x - \frac{c}{2}\int_{\partial M}\left(h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \mu^2\phi^2\right)\mathrm{d}^d x$$

yields the equations of motion

$$-\Box_g\phi + \mu^2\phi = 0 \qquad \text{in } M, \qquad (1)$$

$$-\Box_h\phi + \mu^2\phi = -c^{-1}\partial_\perp\phi \qquad \text{in } \partial M. \qquad (2)$$

Using (1), one may write (2) alternatively as

$$\partial_\perp^2\phi = -c^{-1}\partial_\perp\phi \qquad \text{in } \partial M. \qquad (3)$$

Such boundary conditions are known in the mathematical literature as generalized Wentzell, Wentzell-Feller type, kinematic, or dynamical boundary conditions.

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Different interpretations possible:

- (3) as boundary condition for wave equation (1).
- (1), (2) as wave equations for the bulk and the boundary field, coupled by
  - The bulk field providing a source for the boundary field;
  - The boundary field providing the boundary value of the bulk field.

Strategy

- Write full system as
$$-\partial_t^2 \Phi = \Delta \Phi$$
with Δ a self-adjoint operator on some Hilbert space H.
- Using Δ, rewrite the full system as a first order equation on suitable energy Hilbert spaces for the Cauchy data. This yields well-posedness for smooth initial data with suitable fall-off and global energy estimates.
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Problem: Boundary condition

$$\partial_\perp^2 \phi = -c^{-1} \partial_\perp \phi$$

does not guarantee vanishing of current

$$j = \mathrm{Im}(\bar{\phi} \partial_\perp \phi)$$

through boundary

- The following symplectic form is conserved:

$$\sigma((\phi, \dot{\phi}), (\psi, \dot{\psi})) = \int_{\Sigma} \phi\dot{\psi} - \dot{\phi}\psi + c \int_{\partial\Sigma} \phi\dot{\psi} - \dot{\phi}\psi.$$

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- It is thus natural to consider the Hilbert space

$$H = L^2(\Sigma) \oplus cL^2(\partial\Sigma)$$

with scalar product

$$\langle(\phi_{\mathrm{bk}}, \phi_{\mathrm{bd}}), (\psi_{\mathrm{bk}}, \psi_{\mathrm{bd}})\rangle = \langle\phi_{\mathrm{bk}}, \psi_{\mathrm{bk}}\rangle_{L^2(\Sigma)} + c\langle\phi_{\mathrm{bd}}, \psi_{\mathrm{bd}}\rangle_{L^2(\partial\Sigma)}$$

so that

$$\sigma((\phi, \dot{\phi}), (\psi, \dot{\psi})) = \langle(\bar{\phi}, \bar{\phi}|_{\partial\Sigma}), (\dot{\psi}, \dot{\psi}|_{\partial\Sigma})\rangle - \langle(\bar{\psi}, \bar{\psi}|_{\partial\Sigma}), (\dot{\phi}, \dot{\phi}|_{\partial\Sigma})\rangle.$$

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- We may write the wave equation as

$$-\partial_t^2\Phi = \Delta\Phi = \begin{pmatrix} -\Delta_\Sigma + \mu^2 & 0 \\ c^{-1}\partial_\perp\cdot|_{\partial\Sigma} & -\Delta_{\partial\Sigma}+\mu^2 \end{pmatrix}\begin{pmatrix}\phi_{\mathrm{bk}}\\ \phi_{\mathrm{bd}}\end{pmatrix},$$

where the boundary condition $\phi_{\mathrm{bk}}|_{\partial\Sigma} = \phi_{\mathrm{bd}}$ is encoded in the domain

$$\mathrm{dom}(\Delta) = \left\{(\phi_{\mathrm{bk}},\phi_{\mathrm{bd}}) \in H \mid \phi_{\mathrm{bk}} \in H^2(\Sigma), \phi_{\mathrm{bd}} \in H^2(\partial\Sigma), \phi_{\mathrm{bk}}|_{\partial\Sigma} = \phi_{\mathrm{bd}}\right\}.$$

Proposition

Δ is self-adjoint with spectrum contained in $[\mu^2, \infty)$.

Proof.

For $\Phi \in \mathrm{dom}(\Delta)$, we compute (with $\mu = 0$):

$$\begin{aligned} \langle \Phi, \Delta\Phi \rangle &= -\int_\Sigma \bar{\phi}_{\mathrm{bk}} \Delta_\Sigma \phi_{\mathrm{bk}} + \int_{\partial\Sigma} \bar{\phi}_{\mathrm{bd}} \partial_\perp \phi_{\mathrm{bk}}| - c\bar{\phi}_{\mathrm{bd}} \Delta_{\partial\Sigma} \phi_{\mathrm{bd}} \\ &= \int_\Sigma \partial_i \bar{\phi}_{\mathrm{bk}} \partial_i \phi_{\mathrm{bk}} + c \int_{\partial\Sigma} \partial_j \bar{\phi}_{\mathrm{bd}} \partial_j \phi_{\mathrm{bd}} \geqslant 0. \end{aligned}$$

This entails the bound on the spectrum. The claim on self-adjointness follows similarly by integration by parts: One shows that also on $\mathrm{dom}(\Delta^*)$ the boundary condition $\phi_{\mathrm{bk}}|_{\partial\Sigma} = \phi_{\mathrm{bd}}$ has to be satisfied. □

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Proposition

For smooth Cauchy data

$H^\infty = \bigcap_s H^s$

$$(\phi_0, \phi_1) \in H^\infty(\Sigma) \times H^\infty(\Sigma)$$

such that

$$\partial_\perp^{2k+2}\phi_i|_{\partial\Sigma} = -c^{-1}\partial_\perp^{2k+1}\phi_i|_{\partial\Sigma}, \qquad \forall k \in \mathbb{N},$$

for $i = 0, 1$, *there is a unique smooth solution* $\phi(t)$ *to the wave equation with* $\mu > 0$. *The properties of the Cauchy data are conserved under time evolution. Furthermore, denoting* $\Phi(t) = (\phi(t), \phi(t)|_{\partial\Sigma})$, *we have*

$$\|\partial_t^m \Phi(t)\|_{k+1}^2 + \|\partial_t^{m+1}\Phi(t)\|_k^2 = \|\Phi_0\|_{k+m+1}^2 + \|\Phi_1\|_{k+m}^2.$$

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$$\|\phi\|_k^2 = \|\Delta^{k/2} \phi\|^2$$

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- We consider the bulk and boundary stress-energy tensors

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \tfrac{1}{2} g_{\mu\nu} \left(\partial_\lambda \phi \partial^\lambda \phi + \mu^2 \phi^2 \right),$$
$$T|_{ab} = c \left[\partial_a \phi \partial_b \phi - \tfrac{1}{2} h_{ab} \left(\partial_c \phi \partial^c \phi + \mu^2 \phi^2 \right) \right].$$

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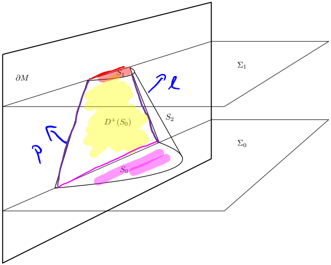
- Both $T_{\mu\nu}$ and $T|_{ab}$ fulfill the dominant energy condition.



We integrate $\nabla^\mu T_{\mu 0}$ and $\nabla^a T|_{a0}$ over $D = D^+(S_0) \cap J^-(\Sigma_1)$ and ∂D:

$$\int_{\partial D} \nabla^a T|_{a0} = \int_{S_1 \cap \partial M} T|_{00} + \int_{S_2 \cap \partial M} p^a T|_{a0} - \int_{S_0 \cap \partial M} T|_{00}$$
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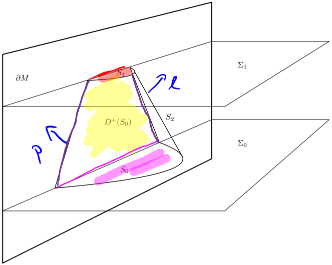

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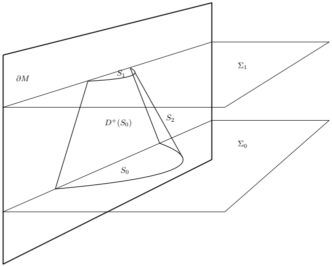

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Proposition

Causal propagation is implied by the local energy estimate

$$\int_{S_1} (\partial_0 \phi)^2 + g^{ij} \partial_i \phi \partial_j \phi + \mu^2 \phi^2 + c \int_{S_1 \cap \partial M} (\partial_0 \phi)^2 + h^{ij} \partial_i \phi \partial_j \phi + \mu^2 \phi^2$$
$$\leqslant \int_{S_0} (\partial_0 \phi)^2 + g^{ij} \partial_i \phi \partial_j \phi + \mu^2 \phi^2 + c \int_{S_0 \cap \partial M} (\partial_0 \phi)^2 + h^{ij} \partial_i \phi \partial_j \phi + \mu^2 \phi^2.$$

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Some comments:

- That $L^2(\Sigma) \oplus L^2(\partial\Sigma)$ is the appropriate space of Cauchy data has been observed by several authors [Feller 57; Ueno 73; Gal, Goldstein & Goldstein 03; ...].
- The global energy estimates for $m = k = 0$ were already known [Vitillaro 15].
- Local energy estimates and thus causal propagation seem to be new.

An example

Consider $\Sigma = \mathbb{R}^d_+$ and a singularity $\delta(t+z)$ infalling to the boundary from the right. The full solution is given by

$$\phi = \delta(t+z) - \delta(t-z) + 2c^{-1}e^{-\frac{t-z}{c}}\theta(t-z).$$

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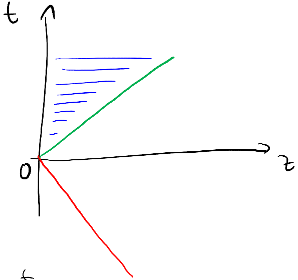

$\phi|_{\partial M} = 2c^{-1}e^{-\frac{t}{c}}G(t)$

Note $c > 0$!!

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Open issue: Propagation of singularities.

Outline

The eigenfunctions

- We consider $\Sigma = \mathbb{R}^d \times [-S, S]$. A basis of eigenfunctions of Δ is

$$\phi_{k,m} = c_m (2\pi)^{-\frac{d-1}{2}} S^{-\frac{1}{2}} e^{ikx} \begin{cases} \cos q_m z & m \text{ even} \\ \sin q_m z & m \text{ odd} \end{cases}$$

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Time-zero fields

- Corresponding to $\{\Phi_{k,m}\}_{k\in\mathbb{R}^{d-1}, m\in\mathbb{N}}$, define the one-particle Hilbert space

$$\mathcal{H}_1 = L^2(\mathbb{R}^{d-1}) \otimes l^2(\mathbb{N}),$$

the corresponding Fock space $\mathcal{F}$, and $a_m(k)$, $a_m(k)^*$ fulfilling

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- For $F = (f_{\rm bk}, f_{\rm bd}) \in \mathrm{dom}(\Delta^{-\frac{1}{4}})$, $G \in \mathrm{dom}(\Delta^{\frac{1}{4}})$, define time zero fields

$$\phi_0(F) = \sum_m \int \frac{\mathrm{d}^{d-1}k}{\sqrt{2\omega_{k,m}}} \left(\langle \bar{F}, \Phi_{k,m}\rangle a_m(k) + \langle \Phi_{k,m}, F\rangle a_m(k)^*\right),$$

$$\pi_0(G) = -i\sum_m \int \mathrm{d}^{d-1}k \frac{\sqrt{\omega_{k,m}}}{\sqrt{2}} \left(\langle \bar{G}, \Phi_{k,m}\rangle a_m(k) - \langle \Phi_{k,m}, G\rangle a_m(k)^*\right).$$

These fulfill the canonical equal time commutation relations, i.e.,

$$[\phi_0(F), \phi_0(F')] = 0, \quad [\pi_0(G), \pi_0(G')] = 0, \quad [\phi_0(F), \pi_0(G)] = i\langle \bar{F}, G\rangle.$$

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- Inserting $F = (0, f_{\rm bd})$, one obtains

$$\phi_0(0, f_{\rm bd}) = \sum_m \int \frac{\mathrm{d}^{d-1}k}{\sqrt{2\omega_{k,m}}} d_m \left(\hat{f}_{\rm bd}(-k) a_m(k) + \hat{f}_{\rm bd}(k) a_m(k)^* \right),$$

which is well defined on a dense domain for $f_{\rm bd} \in L^2(\partial\Sigma)$.

Space-time fields

For space-time fields, we admit $F = (f_{\rm bk}, f_{\rm bd}) \in \mathcal{S}(M) \oplus \mathcal{S}(\partial M)$ and define

$$\phi(F) = \sum_m \int \mathrm{d}t \frac{\mathrm{d}^{d-1}k}{\sqrt{2\omega_{k,m}}} \left(\langle \bar{F}(t), \Phi_{k,m} \rangle e^{-i\omega_{k,m}t} a_m(k) + \langle \Phi_{k,m}, F(t) \rangle e^{i\omega_{k,m}t} a_m(k)^* \right).$$

Proposition

Let $\mu > 0$. The map $F \mapsto \phi(F)$ defines an operator valued distribution on a dense invariant linear domain $\mathfrak{D} \subset \mathcal{F}$ and with F real $\phi(F)$ is essentially self-adjoint. The field ϕ is causal, i.e.,

$$\mathrm{supp}(F) \times \mathrm{supp}(G) \implies [\phi(F), \phi(G)] = 0.$$

There is a unitary representation U of the proper orthochronous Poincaré group $\mathcal{P}_+^\uparrow(d)$, under which the domain $\mathfrak{D}$ is invariant and such that

$$U(a, \Lambda)\phi(F)U(a, \Lambda)^* = \phi(F_{(a,\Lambda)})$$

The vacuum vector $\Omega \in \mathfrak{D}$ is invariant under U, cyclic w.r.t. polynomials of the fields $\phi(f_{\rm bk}, f_{\rm bk}|_{\partial\Sigma})$ or $\phi(0, f_{\rm bd})$, and the spectrum of $P|_{\Omega^\perp}$ is contained in H_μ.

Proof.

- Causality from causal propagation and equal time commutation relations.
- Map to generalized free field ψ on $\mathbb{R}^d$ with ladder operators $a_m(k)^{(*)}$ and masses $\mu_m^2 = \mu^2 + q_m^2$:
$$\phi(F) = \psi(f_F).$$
  Have to define $f_F \in \mathcal{S}$ such that f_F takes prescribed values on the mass shells. Then use standard results on generalized free fields [Jost 65] to obtain self-adjointness, continuity, cyclicity.
- Construction of U trivial.

□

The boundary field

For $f \in \mathcal{S}(\partial M)$, we define the boundary field as

$$\phi_{\mathrm{bd}}(f) = \phi(0, c^{-1} f).$$

Restriction to the two boundaries separately yields

$$\phi^{\pm}_{\mathrm{bd}}(x) = (2\pi)^{-\frac{d-1}{2}} \sum_m (\pm)^m d_m \int \frac{\mathrm{d}^{d-1} k}{\sqrt{2\omega_{k,m}}} \left(e^{-i(\omega_{k,m} t - k\underline{x})} a_m(k) + h.c. \right),$$

i.e., a generalized free field with two-point function

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Let $\mu > 0$ or $d > 2$. Then Δ_+ is a tempered distribution. Its singular support is contained in $\{x \in \mathbb{R}^d | x^2 \leqslant 0\}$ and the projection of its analytic wave front set to the cotangent space is given by $\{k \in \mathbb{R}^d | k^2 \leqslant 0, k^0 > 0\}$. For $d \geqslant 2$, the scaling degree of Δ_+ at coinciding points is $d - 2$.

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- Time-slice property does not hold for ϕ_{bd}. For time-slices larger than $2S$?
- The bound on the analytic wave front set implies that $\phi_{\mathrm{bd}}^{\pm}$ satisfies the Reeh-Schlieder property [Strohmaier, Verch, Wollenberg 02].

The bulk-to-boundary map

Bulk fields $\phi_{\rm bk}$ may be defined as

$$\phi_{\rm bk}(f) = \phi(f, 0)$$

We then have

$$\phi^{\pm}_{\rm bd}(f) = \phi_{\rm bk}(f\delta(z \mp S)),$$
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- $f' \in \mathfrak{D}(\partial_+ M)$ is in general not possible. Maybe for $d = 1$?
- Also works for Wick powers (but locality is lost).

Comparison with other boundary conditions

- Restriction to boundary also possible for Neumann boundary condition.
- Boundary two-point function inherits degree of singularity from the bulk.
- For Dirichlet boundary conditions, one may restrict $\partial_\perp \phi$ to the boundary. Singularity of boundary two-point function is then even stronger than that of the bulk.

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- For Dirichlet boundary conditions, one may restrict $\partial_\perp \phi$ to the boundary. Singularity of boundary two-point function is then even stronger than that of the bulk.
- In the AdS/CFT correspondence for scalar fields, the boundary fields also have anomalous dimensions.
- Holographic image of a bulk observable contained in a local algebra $\mathfrak{A}(\mathcal{O})$ [Rehren 00].

Outline

Summary & Outlook

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- Canonical quantization of the free field.
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Outlook:

- Propagation of singularities.
- Interacting fields.
- Fermions.