Bose Particles in a Box: Convergent Expansion of the Ground State in the Mean Field Limiting Regime

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Motivations and Background

- N Bose (nonrelat.) particles in a finite box of volume $|\Lambda| = L^d$

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- Thermodynamic limit: ρ fixed and $|\Lambda| \to \infty$
- Other regimes: Gross-Pitaevskii

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- Rigorous functional integral (Balaban-Feldman-Knoerrer-Trubowiz)

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4. Outlook

Model

- (Δ with periodic boundary conditions)

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$$\mathcal{F} := \Gamma\left(L^2\left(\Lambda, \mathbb{C}; dx\right)\right) \qquad |\Lambda| = L^d$$

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- $$a(x) = \sum_{\mathbf{j}\in\mathbb{Z}^d} \frac{a_{\mathbf{j}} e^{ik_{\mathbf{j}}\cdot x}}{|\Lambda|^{\frac{1}{2}}},$$

$k_{\mathbf{j}} := \frac{2\pi}{L}\mathbf{j}$, $\mathbf{j} = (j_1, j_2, \ldots, j_d)$, $j_1, j_2, \ldots, j_d \in \mathbb{Z}$

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Assumptions on the two-body potential

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- Strong interaction potential regime: The ratio $\epsilon_{\mathbf{j}} := \frac{k_{\mathbf{j}}^2}{\phi_{\mathbf{j}}}$ is sufficiently small

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Model: Particle Preserving Bogoliubov Hamiltonian

- H is restricted to
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- $H = H^B + V$
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Why studying the three-modes systems?

- In the mean field limiting regime

$$\inf(H - C_N) \to E^B$$

with

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- The thermodynamic limit is already nontrivial for a three-modes system: a large field problem appears

Main results three modes system: Regimes and dimensions

- The Feshbach flow and the construction of the ground state are well defined if $\epsilon_{\mathbf{j}_*} := \frac{k_{\mathbf{j}_*}^2}{\phi_{\mathbf{j}_*}}$ is sufficiently small and for some $\nu > \frac{11}{8}$

$$\epsilon_{\mathbf{j}_*}^{\nu} \geq \frac{1}{N} \quad \Longleftrightarrow \quad \frac{k_{\mathbf{j}_*}^2}{\phi_{\mathbf{j}_*}} > \left(\frac{1}{N}\right)^{\frac{8}{11}}$$

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  - at fixed ρ only if

$$\left[\frac{(2\pi \mathbf{j}_*)^2}{L^2 \phi_{\mathbf{j}_*}}\right]^{\nu} \geq \frac{1}{\rho L^d}$$

$\Rightarrow \quad$ $d \geq 3$ and L large enough

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- Existence of the fixed point if

$$\rho \geq \rho_0 (\frac{L}{L_0})^{3-d}$$

with ρ_0 sufficiently large ($L_0 \equiv 1$)

  - If $d \geq 3 \Rightarrow$ $L < \infty$ can be taken arbitrarily large at fixed (and large) ρ

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- In the mean field limiting regime, $z_* \to E^{Bog}_{\mathbf{j}_*}$ as $N \to \infty$

- For $d = 3$ and $\rho \geq \rho_0$, $z_* \to -\phi_{\mathbf{j}_*}$ as $L \to \infty$

Ideas

- If ψ_{gs} ground state of H, $\langle\sum_{\mathbf{j}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} a_{\mathbf{j}}^* a_{\mathbf{j}}\rangle_{\psi_{gs}}$ stays bounded as $N \to \infty$
  $\Rightarrow$ Conjecture: An effective Hamiltonian in a neighborhood of E_{gs} is a multiple of the projection

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$$\mathscr{F}(K - z) := \mathscr{P}(K - z)\mathscr{P} - \mathscr{P}K\overline{\mathscr{P}}\frac{1}{\overline{\mathscr{P}}(K - z)\overline{\mathscr{P}}}\overline{\mathscr{P}}K\mathscr{P}$$

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- Isospectrality: 1) $\mathscr{F}(K-z)$ is bounded invertible on $\mathscr{P}\mathcal{H}$ if and only if z is in the resolvent set of K (on $\mathcal{H}$); 2) z is an eigenvalue of K if and only if 0 is an eigenvalue of $\mathscr{F}(K-z)$

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- Selection rules of H w.r.t. $\sum_{\mathbf{j}=\pm\mathbf{j}_*} a^*_{\mathbf{j}} a_{\mathbf{j}}$
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- The Rayleigh-Schrödinger expansion of ψ_{gs} is not under control for strong interaction potentials (thermodynamic limit). Can $\overline{\mathscr{P}}$ help to avoiding *small denominator problems*?

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- For the purpose of this talk the Hilbert space $\mathcal{F}^N$ contains only the degrees of freedom $(\mathbf{0}; -\mathbf{j}_*; \mathbf{j}_*)$

Feshbach Projections for $\hat{H}^B$

- $Q^{(i,i+1)} :=$ the projection (in $\mathcal{F}^N$) onto the subspace spanned by the vectors with $N-i$ or $N-i-1$ particles in the modes $\mathbf{j}_*$ and $-\mathbf{j}_*$ $\quad\rightarrow\quad$ the operator $a^*_{\mathbf{j}_*}a_{\mathbf{j}_*} + a^*_{-\mathbf{j}_*}a_{-\mathbf{j}_*}$ has eigenvalues $N-i$ and $N-i-1$ when restricted to $Q^{(i,i+1)}\mathcal{F}^N$

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-
$$\mathcal{F}^N = Q^{(0,1)}\mathcal{F}^N \oplus Q^{(2,3)}\mathcal{F}^N \oplus \cdots \oplus Q^{(N-2,N-1)}\mathcal{F}^N \oplus \{\mathbb{C}\eta\}$$
- $Q^{(>1)}$:= the projection onto the orthogonal complement of $Q^{(0,1)}\mathcal{F}^N$ in $\mathcal{F}^N$ $\rightarrow$ $Q^{(>1)} + Q^{(0,1)} = \mathbf{1}_{\mathcal{F}^N}$
- Iteratively, <u>for i even</u>, $2 \le i \le N-2$, define

$Q^{(>i+1)}$ the projection such that $Q^{(>i+1)} + Q^{(i,i+1)} = Q^{(>i-1)}$

$$Q^{(>N-1)} \equiv |\eta\rangle\langle\eta|$$

Flow of Feshbach Hamiltonians for $\hat{H}^B$

- Define $\mathscr{P}^{(i)} := Q^{(>i+1)}$, $\overline{\mathscr{P}^{(i)}} := Q^{(i,i+1)}$

Flow of Feshbach Hamiltonians for $\hat{H}^B$

- Define $\mathscr{P}^{(i)} := Q^{(>i+1)}$, $\overline{\mathscr{P}^{(i)}} := Q^{(i,i+1)}$
- Starting from $K^B_{-2}(z) := \hat{H}^B - z$

$$\begin{aligned} & K_i^B(z) \\ := \quad & \mathscr{P}^{(i)} K^B_{i-2}(z) \mathscr{P}^{(i)} \\ & - \mathscr{P}^{(i)} K^B_{i-2}(z) \overline{\mathscr{P}}^{(i)} \frac{1}{\overline{\mathscr{P}}^{(i)} K^B_{i-2}(z) \overline{\mathscr{P}}^{(i)}} \overline{\mathscr{P}}^{(i)} K^B_{i-2}(z) \mathscr{P}^{(i)} \end{aligned}$$

General Term

- For i (even)

$$K_i^B(z) := Q^{(>i+1)}(\hat{H}^B - z)Q^{(>i+1)}$$

$$-Q^{(>i+1)} W R_{i,i}^B(z) \sum_{l_i=0}^{\infty} \left[\Gamma_{i,i}^B(z) R_{i,i}^B(z)\right]^{l_i} W^* Q^{(>i+1)}$$

- $$\Gamma_{i+2,i+2}^B(z) :=$$

$$= Q^{(i+2,i+3)} W R_{i,i}^B(z) \sum_{l_i=0}^{\infty} \left[\Gamma_{i,i}^B(z) R_{i,i}^B(z)\right]^{l_i} W^* Q^{(i+2,i+3)}$$

- $$\Gamma_{2,2}^B(z) := Q^{(2,3)} W R_{0,0}^B(z) W^* Q^{(2,3)}$$

Range of the spectral parameter z

- Spectrum of $\hat{H}^B \equiv H^B_{\mathbf{j}_*}$ as $N \to \infty$ (Seiringer):
  - the ground state energy tends to

$$E^B_{\mathbf{j}_*} := -\left[k^2_{\mathbf{j}_*} + \phi_{\mathbf{j}_*} - \sqrt{(k^2_{\mathbf{j}_*})^2 + 2\phi_{\mathbf{j}_*} k^2_{\mathbf{j}_*}}\right]$$

$$E^B_{\mathbf{j}_*} \to -\phi_{\mathbf{j}_*} \quad \text{as} \quad \epsilon_{\mathbf{j}_*} \to 0$$

  - the first excited eigenvalue tends to

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- Question:
  Can we control the flow for $z < E^B_{\mathbf{j}_*} + \sqrt{(k^2_{\mathbf{j}_*})^2 + 2\phi_{\mathbf{j}^*} k^2_{\mathbf{j}_*}}$?

General Term

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$$K_i^B(z) := Q^{(>i+1)}(\hat{H}^B - z)Q^{(>i+1)}$$

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-

$$(R_{i+2,i+2}^B(z))^{\frac{1}{2}} \Gamma_{i+2,i+2}^B(z) (R_{i+2,i+2}^B(z))^{\frac{1}{2}} :=$$

$$= (R_{i+2,i+2}^B(z))^{\frac{1}{2}} W\, R_{i,i}^B(z) \sum_{l_i=0}^{\infty} \left[\Gamma_{i,i}^B(z) R_{i,i}^B(z)\right]^{l_i} W^* (R_{i+2,i+2}^B(z))^{\frac{1}{2}}$$

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Key estimates to control the Feshbach flow

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- key estimate

$$\| \left[R^B_{i,i}(z)\right]^{\frac{1}{2}} W \left[R^B_{i-2,i-2}(z)\right]^{\frac{1}{2}} \| \, \| \left[R^B_{i-2,i-2}(z)\right]^{\frac{1}{2}} W^* \left[R^B_{i,i}(z)\right]^{\frac{1}{2}} \|$$

$$\leq \frac{1}{4(1 + a_{\epsilon_{\mathbf{j}_*}} - \frac{2b_{\epsilon_{\mathbf{j}_*}}}{N-i+1} - \frac{1-c_{\epsilon_{\mathbf{j}_*}}}{(N-i+1)^2})}$$

where $a_{\epsilon_{\mathbf{j}_*}} := \mathcal{O}(\epsilon_{\mathbf{j}_*})$, $b_{\epsilon_{\mathbf{j}_*}} := \mathcal{O}(\sqrt{\epsilon_{\mathbf{j}_*}})$, $c_{\epsilon_{\mathbf{j}_*}} := \mathcal{O}(\epsilon_{\mathbf{j}_*})$

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- $\epsilon_{\mathbf{j}_*} := \frac{k^2_{\mathbf{j}_*}}{\phi_{\mathbf{j}_*}}$ small but, more importantly, $\epsilon^{\nu}_{\mathbf{j}_*} \geq \frac{1}{N}$ for some $\nu > \frac{11}{8}$

  no *small* parameter, i.e., no flow for $\epsilon_{\mathbf{j}_*} \equiv 0$

Key estimates to control the Feshbach flow

- <u>Artificial $\phi_{\mathbf{j}_*}-$dependent Gap</u>

$$R^{B}_{i,i}(z)$$

$$= Q^{(i,i+1)}\frac{1}{Q^{(i,i+1)}(\hat{H}^{B}-z)Q^{(i,i+1)}}Q^{(i,i+1)}$$

- <u>Artificial $\phi_{\mathbf{j}_*}$−dependent Gap</u>

$$
\begin{aligned}
&R^{B}_{i,i}(z)\\
&= Q^{(i,i+1)} \frac{1}{Q^{(i,i+1)}(H^{(0)} + W + W^{*} - z)Q^{(i,i+1)}} Q^{(i,i+1)}
\end{aligned}
$$

Key estimates to control the Feshbach flow

- Artificial $\phi_{\mathbf{j}_*}$ −dependent Gap

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$$H^{(0)} \geq 0 \quad \text{and} \quad z \simeq -\phi_{\mathbf{j}_*}$$

Final Feshbach Hamiltonian, Fixed Point, and GS Energy

- $Q^{(>N-1)} = P_\eta = |\eta\rangle\langle\eta|$, hence

$$K^B_{N-2}(z) = f(z)P_\eta$$

$$f(z) = -z$$

$$-\langle\eta\,,\, W\, R^B_{N-2,N-2}(z)\sum_{l=0}^{\infty}\left[\Gamma^B_{N-2,N-2}(z)R^B_{N-2,N-2}(z)\right]^l W^*\,\eta\rangle$$

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- $f(z)$ is decreasing and there is (only) one point z_* in the interval

$$(-\infty, E^B_{\mathbf{j}_*} + \sqrt{\epsilon_{\mathbf{j}_*}} \sqrt{(k^2_{\mathbf{j}_*})^2 + 2\phi_{\mathbf{j}_*} k^2_{\mathbf{j}_*}})$$

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- $f(z_*) = 0 \quad \Rightarrow \quad z_*$ is the ground state energy of $\hat{H}^B$

GS vector

- Feshbach theory: If φ eigenvector of $\mathscr{F}(K - z_*)$ with eigenvalue 0

$$[\mathscr{P} - \frac{1}{\overline{\mathscr{P}}(K - z_*)\overline{\mathscr{P}}}\overline{\mathscr{P}}K\mathscr{P}]\varphi$$

is eigenvector of K with eigenvalue z_*

GS vector

- Convergent expansion (up to any desired precision)

$$\psi_N^B =$$

$$= \eta$$

$$- \frac{1}{Q^{(N-2,N-1)} K_{N-4}^B(z_*) Q^{(N-2,N-1)}} Q^{(N-2,N-1)} W^* \eta$$

$$- \sum_{j=2}^{N/2} \prod_{r=2j}^{4} \Big[- \frac{1}{Q^{(N-r,N-r+1)} K_{N-r-2}^B(z_*) Q^{(N-r,N-r+1)}} W^*_{N-r,N-r+2} \Big] \times$$

$$\times \frac{1}{Q^{(N-2,N-1)} K_{N-4}^B(z_*) Q^{(N-2,N-1)}} Q^{(N-2,N-1)} W^* \eta$$

where $W^*_{N-r,N-r+2} := Q^{(N-r,N-r+1)} W^* Q^{(N-r+2,N-r+3)}$

GS vector

- Fix $\epsilon_{\mathbf{j}_*}$. Then, $\forall \xi > 0$ $\exists \bar{j}_\xi$, N_ξ such that $\forall N > N_\xi$ with the property

$$\psi_N^B =$$

$$= \eta$$

$$-\frac{1}{Q^{(N-2,N-1)} K_{N-4}^B(z_*) Q^{(N-2,N-1)}} Q^{(N-2,N-1)} W^* \eta$$

$$-\sum_{j=2}^{\bar{j}_\xi} \prod_{r=2j}^{4} \left[-\frac{1}{Q^{(N-r,N-r+1)} K_{N-r-2}^B(z_*) Q^{(N-r,N-r+1)}} W_{N-r,N-r+2}^* \right] \times$$

$$\times \frac{1}{Q^{(N-2,N-1)} K_{N-4}^B(z_*) Q^{(N-2,N-1)}} Q^{(N-2,N-1)} W^* \eta$$

$$+\mathcal{O}(\xi)$$

Complete Hamiltonian: Control of the cubic and quartic terms

- New first step in each Feshbach flow

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- *Short range property* of the interaction Hamiltonian in the particle states occupation numbers

Complete Hamiltonian: Control of the cubic and quartic terms

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- Semigroup property of the Feshbach map

Outlook / Gross Pitaevskii limit and beyond

- N Bose (nonrelat.) particles in a finite box of volume $|\Lambda| = 1$

$$H = -\sum_i \Delta_i^{(x)} + g\, N^2 \sum_{i<j} \phi(N(x_i - x_j))$$

with $N \to +\infty$

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- Three-modes Hamiltonian

$$H_{\mathbf{j}}^B = \sum_{\pm\mathbf{j}} (N^2 k_{\mathbf{j}}^2 + g\frac{\phi_{\mathbf{j}}}{N} a_{\mathbf{0}}^* a_{\mathbf{0}}) a_{\mathbf{j}}^* a_{\mathbf{j}} + g\frac{\phi_{\mathbf{j}}}{N} \left\{ a_{\mathbf{0}}^* a_{\mathbf{0}}^* a_{\mathbf{j}} a_{-\mathbf{j}} + a_{\mathbf{j}}^* a_{-\mathbf{j}}^* a_{\mathbf{0}} a_{\mathbf{0}} \right\}$$

where $k_{\mathbf{j}}^2 \gtrsim N^{-2}$

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with $N, g \to +\infty$

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- Three-modes Hamiltonian

$$H_{\mathbf{j}}^B = g\phi_{\mathbf{j}}\Big[\sum_{\pm \mathbf{j}}\Big(\frac{N^2 k_{\mathbf{j}}^2}{g\phi_{\mathbf{j}}} + \frac{1}{N} a_{\mathbf{0}}^* a_{\mathbf{0}}\Big) a_{\mathbf{j}}^* a_{\mathbf{j}} + \frac{1}{N}\left\{a_{\mathbf{0}}^* a_{\mathbf{0}}^* a_{\mathbf{j}} a_{-\mathbf{j}} + a_{\mathbf{j}}^* a_{-\mathbf{j}}^* a_{\mathbf{0}} a_{\mathbf{0}}\right\}\Big]$$

where $k_{\mathbf{j}}^2 \gtrsim N^{-2} \quad \Rightarrow \quad \frac{N^2}{g}\frac{k_{\mathbf{j}}^2}{\phi_{\mathbf{j}}} > N^{-\frac{8}{11}} \quad$ for $\quad g \lesssim N^{\frac{8}{11}}$

THANK YOU

Key estimates to control the Feshbach flow

- Control of the sequence

$$X_{i+2} := 1 - \frac{1}{4(1 + a_{\epsilon_{\mathbf{j}*}} - \frac{b_{\epsilon_{\mathbf{j}*}}}{N-i+1} - \frac{1-c_{\epsilon_{\mathbf{j}*}}}{(N-i+1)^2})} \frac{1}{X_i}$$

$X_0 \equiv 1$ and, $0 \le i \le N-2$ and even