Wedge local fields in integrable models with bound states

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Outline

Integrable models

- Bosons (no spin, with mass $\mu > 0$) in 1+1 dimensional spacetime
- Two-momentum and rapidity:

$$p = p(\theta) = \mu(\cosh\theta, \sinh\theta)$$

- Two-particle scattering allows exchange of phase factor
  - two-particle scattering matrix $S(\theta_1 - \theta_2)$.
- multi-particle scattering matrix – product of two-particle scattering matrices (“factorizing S matrix”).
- The two-particle scattering function S is
  - a meromorphic function in the strip $0 < \operatorname{Im}\theta < \pi$
  - with certain symmetry properties,
  - $S = 1$: free field; $S = -1$: Ising model, other examples: sinh-Gordon model, Bullough-Dodd model.

Task: Given a function S, construct a corresponding quantum field theory.

Outline

Deformed Hilbert space and deformed fields

The theory is constructed as a deformation of a free field:

- Zamolodchikov-Faddeev algebra (elements $z(\theta)$, $z^\dagger(\theta)$):

$$
\begin{aligned}
z(\theta_1)z(\theta_2) &= S(\theta_1 - \theta_2)\, z(\theta_2)z(\theta_1)\,, \\
z^\dagger(\theta_1)z^\dagger(\theta_2) &= S(\theta_1 - \theta_2)\, z^\dagger(\theta_2)z^\dagger(\theta_1)\,, \\
z(\theta_1)z^\dagger(\theta_2) &= S(\theta_2 - \theta_1)\, z^\dagger(\theta_2)z(\theta_1) + \delta(\theta_1 - \theta_2)\cdot \mathbf{1}.
\end{aligned}
$$

  These act on an "S-symmetric" Fock space.

- Representation of the Poincaré group, including the space-time reflections J.
- Define

$$\phi(x) := \int d\theta \left(e^{ip(\theta)\cdot x} z^\dagger(\theta) + e^{-ip(\theta)\cdot x} z(\theta) \right).$$

- This field is not local:

$$[\phi(x), \phi(y)] \neq 0 \text{ even if } x \text{ spacelike separated from } y.$$

Local observables

- But, with $\phi'(x) := U(j)\phi(-x)U(j)$:

$$[\phi(x), \phi'(y)] = 0 \text{ if } x \text{ spacelike separated to the left of } y.$$

This assumes that S is analytic in the "physical strip" $0 < \operatorname{Im} \zeta < \pi$.

- Interpretation: $\phi(x)$ is localized in the wedge region $W_L + x$, and $\phi'(y)$ is localized in the wedge region $W_R - y$.
- Further wedge-local observables by relative locality / associated von Neumann algebras:

$$\mathcal{A}(W_L + x) = \{\exp i\phi(f) \,|\, \operatorname{supp} f \subset W_L + x\}''$$

- Observables localized in bounded regions are obtained as intersections of von Neumann algebras

$$\mathcal{A}(\mathcal{O}) := \mathcal{A}(W_L + x) \cap \mathcal{A}(W_R - y) \quad \text{where} \quad \mathcal{O} = W_L + x \cap W_R - y$$

- Result (Lechner 2006): Such observables exist for a large class of S.

Outline

Bound states

Now suppose that $S(\theta)$ has poles in the physical strip $0 < \operatorname{Im}\theta < \pi$.

- Physically these poles correspond to "bound states", that is the "fusion" of two bosons.
  - Simplification: only one type of particle; two bosons of equal type fuse to form another boson of the same type.
- The momenta of the particles are related by $p(\theta_1) + p(\theta_2) = p(\theta_b)$, where θ_1, θ_2 and θ_b are the (complex) rapidities of the two fusing bosons and of the bound particle, respectively.
- The difference of the rapidities of the fusing bosons is the position of the pole on the rapidity complex plane: $\theta_1 - \theta_2 = i\lambda$ $(0 < \lambda < \pi)$.
  - If the particles have all equal masses, this is fulfilled if and only if $\theta_1 = \theta + \frac{i\pi}{3}$, $\theta_2 = \theta - \frac{i\pi}{3}$ and $\theta_b = \theta$ (that is, $\lambda = \frac{2\pi}{3}$.)

Bound states

- In Lechner's work, the commutator $[\phi'(f), \phi(g)]$ is seen to be zero by shifting an integral contour from $\mathbb{R}$ to $\mathbb{R} + i\pi$.
  - But due to the residue of S at the pole $\frac{2\pi i}{3}$, this is no longer true.
- We need to modify ϕ to get a wedge-local expression.

Outline

Wedge-local model with bound states

The properties of the two-particle scattering function are

- Unitarity: $S(-\theta) = \overline{S(\theta)} = S(\theta)^{-1}$.
- Crossing symmetry: $S(i\pi - \theta) = S(\theta)$.
- Bootstrap equation: $S(\theta) = S(\theta + \frac{i\pi}{3})S(\theta - \frac{i\pi}{3})$.

Example for such a function S: Bullough-Dodd model.

$$S(\zeta, B) = f_{\frac{2}{3}}(\zeta) f_{\frac{B}{3}-\frac{2}{3}}(\zeta) f_{-\frac{B}{3}}(\zeta),$$

where

$$f_a(\zeta) := \frac{\tanh \frac{1}{2}(\zeta + i\pi a)}{\tanh \frac{1}{2}(\zeta - i\pi a)}, \quad 0 < B < 1.$$

Wedge-local model with bound states

We introduce, on the S-symmetric Fock space, the "bound state operator". On the single-particle Hilbert space $\mathcal{H}_1$:

$$\mathrm{Dom}(\chi_1(f)) :=$$

$$\{\xi \in \mathcal{H}_1 : \xi(\theta) \text{ has an } L^2\text{-bounded analytic continuation to } \theta - \frac{i\pi}{3}\},$$

$$(\chi_1(f)\xi)(\theta) := \sqrt{2\pi|R|} f^+\left(\theta + \frac{i\pi}{3}\right) \xi\left(\theta - \frac{i\pi}{3}\right),$$

where $R := \mathrm{res}_{\zeta=\frac{2\pi i}{3}} S(\zeta)$. Note: $\chi_1(f)$ realizes the idea that the state of one elementary particle ξ is fused with f^+ into the same species of particle.

On the S-symmetric Fock space: (P projector onto this space)

$$\chi_n(f) := nP_n(\chi_1(f) \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1})P_n,$$

$$\chi(f) = \bigoplus_{n=0}^{\infty} \chi_n(f).$$

Wedge-local model with bound states

Note: As a consequence of crossing symmetry, the two-particle scattering function has another pole at $\theta' = i\pi - \frac{2i\pi}{3} = \frac{i\pi}{3}$ with residue

$$R' := \mathrm{res}_{\zeta = \frac{i\pi}{3}} S(\zeta).$$

As a consequence of the properties of S, one finds that $R' = -R$ and that R is purely imaginary.

Wedge-local model with bound states

We define a new field

$$\tilde{\phi}(f) = \phi(f) + \chi(f),$$

where

$$\phi(f) = z^\dagger(f_+) + z(f_-)$$

$(f_\pm(\theta) = \int d\theta \, e^{\pm ip(\theta)\cdot x} f(x))$.
We can introduce the reflected field as $\tilde{\phi}'(g) := J\tilde{\phi}(jg)J$.

Outline

Weak wedge commutativity

Consider the following linear space of vectors: $\Psi \in \mathrm{Dom}(\tilde{\phi}(f)) \cap \mathrm{Dom}(\tilde{\phi}'(g))$ such that $\prod_j S\left(\theta - \theta_j + \frac{\pi i}{3}\right) \Psi_n(\theta, \theta_1, \cdots, \theta_{n-1})$ and $\prod_j S\left(\theta - \theta_j + \frac{2\pi i}{3}\right) \Psi_n(\theta, \theta_1, \cdots, \theta_{n-1})$ have $L^2(\mathbb{R}^{n-1})$-valued bounded analytic continuations in θ to $\theta \pm \epsilon i$ for some $\epsilon > 0$.

Theorem

Let f and g be real test functions supported in W_L and W_R, respectively. Then, for each Φ, Ψ in the linear space above, it holds that

$$\langle \tilde{\phi}(f)\Phi, \tilde{\phi}'(g)\Psi \rangle = \langle \tilde{\phi}'(g)\Phi, \tilde{\phi}(f)\Psi \rangle.$$

- Note: It is the commutator of χ with its reflected operator χ' that cancels the contribution of the residues coming from the commutator between ϕ and ϕ', mentioned before.

Outlooks

- The fields $\tilde{\phi}(f)$ and $\tilde{\phi}'(g)$ do not preserve their domains, especially one cannot iterate them on the vacuum more than once.
- The Reeh-Schlieder property does not hold for polynomials of these fields. Instead, an argument can be drawn if we assume the existence of nice self-adjoint extensions.
- The field $\tilde{\phi}(f)$ is a polarization-free generator, but non-temperate.
- To show: $\tilde{\phi}(f)$ and $\tilde{\phi}'(g)$ have self-adjoint extensions and they strongly commute. (see talk by Y. Tanimoto)
- Apply Haag-Ruelle scattering theory. (work in progress)
- Construction of Haag-Kastler nets: prove the modular nuclearity condition for the associated wedge-local nets and for separations of wedges larger than the minimal distance. (work in progress)

Outline

Other models

Other interesting models with poles in the physical strip contain more than one type of particle.

- $Z(N)$ model: has particles labelled $1, \ldots, N-1$, where $N-j$ is the anti-particle of j ($\bar{j} = N-j$) with fusion rules $(jk) = j+k \bmod N$.
- Sine-Gordon model: has particles $s, \bar{s}$ ("soliton", "anti-soliton") and a finite number of bound states of s and $\bar{s}$ called b_k ("breathers").

$$(s, \bar{s}) = b_k, \qquad (s, b_k) = s, \qquad (\bar{s}, b_k) = \bar{s},$$
$$(b_k, b_l) = b_{k+l}, \qquad (b_{k+l}, b_k) = b_l.$$

Other models

What changes in these models:

- There is a larger single particle space, i.e., there are several types of creators and annihilators: $z_\alpha, z_\alpha^\dagger$.
- The two-particle scattering function is a matrix $S_{\gamma\delta}^{\alpha\beta}(\zeta)$.
- Fusion angles can be more complicated $\theta_{\alpha\beta}^\gamma = \theta_{(\alpha\beta)}^\gamma + \theta_{(\beta\alpha)}^\gamma$.
- Wedge local fields and associated local nets in the case without poles in the physical strip have been worked out by Lechner-Schützenhofer and Alazzawi

$$\phi(f) = z_\alpha(Jf_{-\alpha}) + z_\alpha^\dagger(f_{+\alpha}).$$

- In the case with poles in the physical strip, the proof of wedge-locality requires a new form of the bootstrap equation:

$$S_{\gamma\nu}^{\mu\hat{\gamma}}(\zeta)\eta_{\alpha\beta}^{\gamma} = \eta_{\hat{\alpha}\hat{\beta}}^{\hat{\gamma}} S_{\alpha k}^{\mu\hat{\alpha}}(\zeta + i\theta_{(\alpha\beta)}^\gamma) S_{\beta\nu}^{k\hat{\beta}}(\zeta - i\theta_{(\beta\alpha)}^\gamma),$$

where the matrix η is related to the residue of S.

Other models

…and the Yang-Baxter equation:

$$S^{\alpha\beta}_{\beta'\alpha'}(\theta)S^{\alpha'\gamma}_{\gamma'\alpha''}(\theta+\theta')S^{\beta'\gamma'}_{\gamma''\beta''}(\theta') = S^{\beta\gamma}_{\gamma'\beta'}(\theta')S^{\alpha\gamma'}_{\gamma''\alpha'}(\theta+\theta')S^{\alpha'\beta'}_{\beta''\alpha''}(\theta).$$

together with a new χ, acting on $\mathcal{H}_1$ as

$$(\chi_1(f)\xi)_\gamma(\theta) := \sum_{\alpha\beta} \eta^\gamma_{\alpha\beta} f^+_\alpha(\theta + i\theta_{(\alpha\beta)})\xi_\beta(\theta - i\theta_{(\beta\alpha)}).$$

- What we have so far:
  - In the $Z(N)$-Ising model certain components of the fields $\tilde{\phi}(f)$ and $\tilde{\phi}'(g)$ weakly commute on a dense domain. (work in progress)
  - In the sine-Gordon model, if we restrict ourselvies to soliton, anti-soliton and only one breather, we can prove weak commutativity. Additionally, we can show in the presence of many breathers (but not of $s, \bar{s}$), that the commutator of the fields $\tilde{\phi}(f)$ and $\tilde{\phi}'(g)$ restricted to the particle of type b_1, and sandwiched between states corresponding to particles of type b_k and b_{k+2}, also vanishes. (work in progress)

Summary and outlook

- We have investigated integrable models where the two-particle scattering function has poles in the physical strip.
- We have modified Lechner's definition of wedge-local field by adding an extra term "χ".
- In some models, e.g. Bullough-Dodd, this again yields a wedge-local quantity.
- In other models, e.g. $Z(N)$ and sine-Gordon, we have obtained partial results.
- Operator theoretic properties of $\tilde{\phi}$ are a difficult issue and are under investigation (see talk by Y. Tanimoto).
- Questions concerning the construction of Haag-Kastler nets (modular nuclearity condition) and the application of Haag-Ruelle scattering theory for scalar S-matrices are work in progress.