Spectral analysis of a model for quantum friction

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Workshop on "Quantum Field Theory: Infrared Problems and Constructive Aspects",
München, October 2015

Joint work with S. De Bièvre and B. Schubnel

The model

Linear friction

Linear Friction

Many classical systems – e.g. an electron in a metal, a particle in a viscous medium – obey an effective equation of motion of the form

$$m\ddot{q}(t) = -\gamma \dot{q}(t) - \nabla V(q(t)) \tag{1}$$

where

- $q(t) \in \mathbb{R}^d$ is the position of the system
- m is the mass of the system
- $\gamma > 0$ is the friction coefficient
- V is an external potential

In particular, $V = 0 \implies \dot{q}(t)$ converges exponentially fast to 0

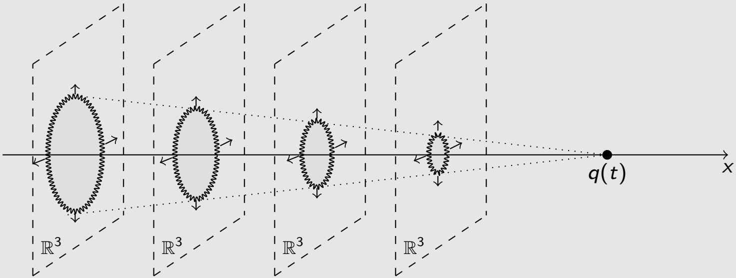
Interaction with the environment

- (1) $=$ effective equation of motion
- Friction force due to the energy lost by the system, transferred to the environment
- More fundamental approach : model describing both the system and its environment with total energy conserved

A classical Hamiltonian model

[Bruneau, De Bièvre, 2002]

Particle of position $q(t) \in \mathbb{R}^d$ (mass $m = 1$, no external potential) coupled to independent scalar vibration fields $\psi(x, y, t) \in \mathbb{R}$ at each point $x \in \mathbb{R}^d$ ($y \in \mathbb{R}^3$ accounts for the position variable in the "propagation space" of the fields)



Equations of motion

$$\partial_t^2 \psi(x, y, t) - c^2 \Delta_y \psi(x, y, t) = -\rho_1(x - q(t))\rho_2(y)$$

$$\ddot{q}(t) = -\int_{\mathbb{R}^{d+3}} \rho_1(x - q(t))\rho_2(y)(\nabla_x \psi)(x, y, t)\, dxdy$$

A classical Hamiltonian model II

Equations of motion

$$\partial_t^2 \psi(x,y,t) - c^2 \Delta_y \psi(x,y,t) = -\rho_1(x-q(t))\rho_2(y)$$
$$\ddot{q}(t) = -\int_{\mathbb{R}^{d+3}} \rho_1(x-q(t))\rho_2(y)(\nabla_x \psi)(x,y,t)\, dxdy$$

Should be compared with

Classical Nelson model

Classical particle coupled to a scalar wave field

$$\partial_t^2 \psi(x,t) - c^2 \Delta_x \psi(x,t) = -\rho_1(x-q(t))$$
$$\ddot{q}(t) = -\int_{\mathbb{R}^{d}} \rho_1(x-q(t))(\nabla_x \psi)(x,t)\, dx$$

Other related classical models

See [Komech, Spohn, 1998], [Komech, Kunze, Spohn, 1998]

A classical Hamiltonian model III

Equations of motion

$$\partial_t^2 \psi(x,y,t) - c^2 \Delta_y \psi(x,y,t) = -\rho_1(x-q(t))\rho_2(y)$$

$$\ddot{q}(t) = -\int_{\mathbb{R}^{d+3}} \rho_1(x-q(t))\rho_2(y)(\nabla_x \psi)(x,y,t)\, dxdy$$

Assumptions

- $\rho_1 \in \mathcal{S}(\mathbb{R}^d)$, positive, radial
- $\rho_2 \in \mathcal{S}(\mathbb{R}^3)$, positive, radial and $\hat{\rho}_2(k) \neq 0 \quad \forall k \in \mathbb{R}^3$

Results [Bruneau, De Bièvre 2002]

For a large class of initial data, and for c large enough, the particle stops exponentially fast,

$$|q(t) - q_\infty| \leq Ce^{-\tilde{\gamma} t}, \quad t \geq 0$$

Hilbert space

Hilbert space for the particle and the field

$$\mathcal{H} = L^2(\mathbb{R}^d) \otimes \mathcal{F}_s(L^2(\mathbb{R}^{d+3}))$$

Symmetric Fock space

- $$\mathcal{F}_s(L^2(\mathbb{R}^{d+3})) = \bigoplus_{n\geq 0} \mathcal{F}_s^{(n)}$$
  where
  $$\mathcal{F}_s^{(0)} := \mathbb{C}, \quad \mathcal{F}_s^{(n)} := L_s^2(\mathbb{R}^{(d+3)n})$$
- Creation and annihilation operators denoted by $a^*(\xi, k)$, $a(\xi, k)$ (momentum variables) satisfy the canonical commutation relations
  $$[a(\xi,k), a^*(\xi',k')] = \delta(\xi - \xi')\delta(k - k'),$$
  $$[a^\#(\xi,k), a^\#(\xi',k')] = 0$$

Total Hamiltonian

Total Hamiltonian acting on $\mathcal{H} = L^2(\mathbb{R}^d) \otimes \mathcal{F}_s(L^2(\mathbb{R}^{d+3}))$

$$H := \frac{-\Delta_q}{2} \otimes \mathbb{1} + \mathbb{1} \otimes H_f + g H_I,$$

where

- Hamiltonian for the particle : $-\Delta_q/2$
- Hamiltonian for the field :

$$H_f = \int_{\mathbb{R}^{d+3}} |k| a^*(\xi, k) a(\xi, k) d\xi dk$$

- Interaction Hamiltonian :

$$H_I := \int_{\mathbb{R}^{d+3}} \Big(e^{-iq\cdot\xi} |k|^{\mu} \hat{\rho}_1(|\xi|) \hat{\rho}_2(|k|) a^*(\xi, k) + e^{iq\cdot\xi} |k|^{\mu} \overline{\hat{\rho}_1(|\xi|) \hat{\rho}_2(|k|)} a(\xi, k) \Big) d\xi dk$$

- $g \in \mathbb{R}$: coupling constant
- $\mu \geq -1/2$: infrared regularization
- $\rho_1 \in \mathcal{S}(\mathbb{R}^d)$, $\rho_2 \in \mathcal{S}(\mathbb{R}^3)$

Properties

Self-adjointness

For all $g \in \mathbb{R}$ and $\mu > -1$, H is a self-adjoint operator with domain

$$\mathcal{D}(H) = \mathcal{D}(H_0),$$

where $H_0 := H|_{g=0}$

Translation invariance

- Let
$$P_f = \int_{\mathbb{R}^{d+3}} \xi a^*(\xi,k) a(\xi,k) d\xi dk$$
Then
$$[(-i\nabla_q \otimes \mathbb{1} + \mathbb{1} \otimes P_f)_j, H] = 0, \quad j = 1, \dots, d$$
- Unitary transformation $U : \mathcal{H} \to \int_{\mathbb{R}^d}^{\oplus} \mathcal{H}_p dp$, $\mathcal{H}_p = \mathcal{F}_s(L^2(\mathbb{R}^{d+3}))$, such that
$$UHU^* = \int_{\mathbb{R}^d}^{\oplus} H(p) dp$$

The fiber Hamiltonian

Fiber Hamiltonian acting on $\mathcal{F}_s(L^2(\mathbb{R}^{d+3}))$

- For all $p \in \mathbb{R}^d$,

$$H(p) := (p - P_f)^2/2 + H_f + g H_{I,0},$$

- Interaction Hamiltonian at fixed total momentum :

$$H_{I,0} := \int_{\mathbb{R}^{d+3}} |k|^{\mu} \big(\hat\rho_1(|\xi|)\hat\rho_2(|k|) a^*(\xi,k) + \overline{\hat\rho_1(|\xi|)\hat\rho_2(|k|)} a(\xi,k) \big) d\xi dk$$

- For all $p \in \mathbb{R}^d$, $g \in \mathbb{R}$ and $\mu > -1$, $H(p)$ is a self-adjoint operator with domain

$$\mathcal{D}(H(p)) = \mathcal{D}(H_f) \cap \mathcal{D}(P_f^2)$$

Spectrum of the non-interacting Hamiltonian

$$\sigma(H_0(p)) = \sigma_{\mathrm{ess}}(H_0(p)) = \sigma_{\mathrm{ac}}(H_0(p)) = [0,\infty),$$
$$\sigma_{\mathrm{pp}}(H_0(p)) = \{p^2/2\}, \qquad \sigma_{\mathrm{sc}}(H_0(p)) = \emptyset$$

Moreover $p^2/2$ is a simple eigenvalue associated to the vacuum $\Omega \in \mathcal{F}_s(L^2(\mathbb{R}^{d+3}))$

Main results

Theorem [De Bièvre, Faupin, Schubnel]

i) Suppose that $\mu > -1$. For all $g \in \mathbb{R}$, there exists $E_g \leq 0$ such that

$$\sigma(H(p)) = \sigma_{\rm ess}(H(p)) = [E_g, \infty),$$

for all $p \in \mathbb{R}^d$. In particular, $E_g = \inf \sigma(H(p))$ does not depend on p

ii) Suppose that $\mu > -1/2$. There exists $g_c = g_c(\mu) > 0$ such that, for all $0 \leq |g| \leq g_c$,

$$H(0) \text{ admits a unique ground state},$$

namely E_g is a simple eigenvalue of $H(0)$

ii') Suppose that $-1 < \mu \leq -1/2$ and that $\hat\rho_1(0) \neq 0$, $\hat\rho_2(0) \neq 0$. For all $p \in \mathbb{R}^d$ and $g \in \mathbb{R}$,

$$H(p) \text{ does } \textit{not} \text{ have a ground state}$$

iii) Suppose that $\mu > 1/2$. There exists $g_c = g_c(\mu) > 0$ such that, for all $0 \leq |g| \leq g_c$,

$$\sigma_{\rm pp}(H(0)) = \{E_g\}, \quad \sigma_{\rm ac}(H(0)) = [E_g, \infty), \quad \sigma_{\rm sc}(H(0)) = \emptyset.$$

Suppose in addition that $\hat\rho_1$ and $\hat\rho_2$ do not vanish and let ν_1, ν_2 be such that $0 < \nu_1 < \nu_2$. Then there exists $g_c = g_c(\mu, \nu_1, \nu_2) > 0$ such that, for all $0 < |g| \leq g_c$ and $p \in \mathbb{R}^d$, $|p| \in (\nu_1, \nu_2)$,

$$\sigma_{\rm pp}(H(p)) = \emptyset, \quad \sigma_{\rm ac}(H(p)) = [E_g, \infty), \quad \sigma_{\rm sc}(H(p)) = \emptyset.$$

In particular, for $|p| \in (\nu_1, \nu_2)$, $H(p)$ does not have a ground state and the unperturbed eigenvalue $p^2/2$ disappears as the coupling is turned on

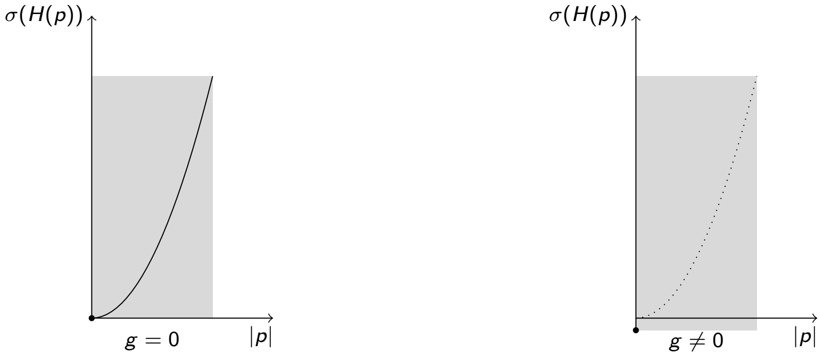


FIGURE: Grey : absolutely continuous spectrum
If the coupling constant $g = 0$, $\inf \sigma(H_0(p)) = 0$ for all p ; $p^2/2$ is a simple eigenvalue of $H(p)$
If the coupling constant $g \neq 0$, $\inf \sigma(H(p)) = E_g < 0$ for all p ; E_g is an eigenvalue if and only if $p = 0$; If $p \neq 0$, the spectrum is purely absolutely continuous

Ingredients of the proof

Location of the spectrum

Theorem

Let $\mu > -1$ and $g \in \mathbb{R}$. There exists $E_g \leq 0$ such that

$$\sigma(H(p)) = [E_g, \infty),$$

for all $p \in \mathbb{R}^d$

Idea

- Localization techniques ([Derezinski, Gérard, 1998])
- General idea : To any state φ with total momentum p, sufficiently localized in x-space, we can add a one-particle state $a^*(f)\Omega$, with f localized near infinity in x-space, such that $a^*(f)\Omega$ has a momentum close to $\xi = -p$ and an energy close to $|k| = 0$. Then $a^*(f)\varphi$ (which can be defined in a proper sense) has an energy arbitrary close to φ and a momentum arbitrary close to 0.
$\Longrightarrow \inf \sigma(H(0)) \leq \inf \sigma(H(p))$
- Difficulty : estimate localization errors, in particular control the number of particles in the minimizing sequence

Existence of a ground state for $H(0)$

Theorem

Let $\mu > -1/2$. There is $g_c > 0$ such that for all $|g| \leq g_c$, $H(0)$ has a ground state

Idea

- Spectral renormalization group ([Bach, Fröhlich, Sigal 1998])
- Iterative version introduced in ([Ballesteros, Faupin, Fröhlich, Schubnel 2015])
- Important new feature : control first and *second* derivatives of Wick monomial kernels. Use rotation invariance

Remark

- [Gérard 2000], [Griesemer, Lieb, Loss 2001] : compactness argument (not satisfied here)
- [Pizzo 2003] : iterative perturbation theory (not applicable here)

Infrared problem : absence of ground state for $\mu \leq -1/2$

Theorem

Suppose that $-1 < \mu \leq -1/2$ and that $\hat{\rho}_1(0) \neq 0$, $\hat{\rho}_2(0) \neq 0$. For all $p \in \mathbb{R}^d$ and $g \in \mathbb{R}$,

$H(p)$ does *not* have a ground state

Idea

- Argument by contradiction
- Use the pull-through formula
- Adapt a simple argument of [Derezinski, Gérard 2004]

Absolutely continuous spectrum, Local decay

Theorem

Suppose that $\mu > 1/2$. There exists $g_c > 0$ such that, for all $|g| \leq g_c$ and $p \in \mathbb{R}^d$, the following holds : Let $J \subset [E_g, \infty)$ be a compact interval such that $\sigma_{\mathrm{pp}}(H(p)) \cap J = \emptyset$. Then

$$\sup_{z \in S} \| \langle A \rangle^{-s} (H(p) - z)^{-1} \langle A \rangle^{-s} \| < \infty,$$

for any $1/2 < s \leq 1$, with $A = \mathrm{d}\Gamma(ik \cdot \nabla_k / |k| + h.c.)$, $\langle A \rangle = (1 + A^* A)^{1/2}$ and

$$S = \{ z \in \mathbb{C}, \mathrm{Re}(z) \in J, 0 < |\mathrm{Im}(z)| \leq 1 \}.$$

In particular, the spectrum of $H(p)$ in J is purely absolutely continuous. Moreover,

$$\left\| \langle A \rangle^{-s} e^{-itH(p)} \chi(H(p)) \langle A \rangle^{-s} \right\| \lesssim t^{-s+\frac{1}{2}}, \quad t \to \infty,$$

for any $1/2 < s \leq 1$ and $\chi \in \mathrm{C}_0^\infty(J; \mathbb{R})$

Idea

- Mourre's commutator method [Mourre 1981]
- Extension with a non self-adjoint conjugate operator, and a first commutator not controllable by the Hamiltonian [Georgescu, Gérard, Møller 2004]

Absence of eigenvalues for $H(p)$, $p \neq 0$, $g \neq 0$

Theorem

Let $\mu > 1/2$ and ν_1, ν_2 be such that $0 < \nu_1 < \nu_2$. There exists $g_c = g_c(\mu, \nu_1, \nu_2) > 0$ such that, for all $0 < |g| \leq g_c$ and $p \in \mathbb{R}^d$, $|p| \in (\nu_1, \nu_2)$,

$$\sigma_{\mathrm{pp}}(H(p)) = \emptyset$$

Idea

- Mourre's commutator method [Georgescu, Gérard, Møller 2004]
- Fermi Golden Rule criterion ([Hunziker, Sigal 2000], [Faupin, Møller, Skibsted 2011])

$$\Pi_\Omega H_{I,0} \mathrm{Im}\big((H_0(p) - p^2 - i0^+)^{-1} \bar{\Pi}_\Omega\big) H_{I,0} \Pi_\Omega \geq \mathrm{c}(p) \Pi_\Omega,$$

where Π_Ω is the projection onto the Fock vacuum and $\bar{\Pi}_\Omega := \mathbb{1} - \Pi_\Omega$

Thank you !