A solvable quantum field theory in 4 dimensions

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(based on joint work with Raimar Wulkenhaar,
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Introduction

Prove that a non-trivial toy model for a quantum field theory on $\mathbb{R}^4$ exists and satisfies [O-S, Wightman].

- Φ^4_4 is renormalizable, $\Phi^4_{4+\epsilon}$ trivial
- Φ^4_4 on Moyal space is nonrenormalizable due to IR/UV mixing
- $(\Phi^4_4)_{modified}$ on Moyal space is renormalizable (HG+R Wulkenhaar, 2004)
- β function is perturbative zero (Rivasseau et al, 2006)
- Ward identities allow to decouple SD equs.

$\rightarrow$ Can we construct it?

Regularisation of $\lambda\phi_4^4$ on noncommutative space

$$S[\phi]=\int_{\mathbb{R}^4} dx\Big(\frac{1}{2}\phi\;\;(-\Delta \qquad\qquad\qquad\qquad + \mu^2\quad)\phi + \frac{\lambda}{4}\phi\;\;\phi\;\;\phi\;\;\phi\Big)(x)$$

Regularisation of $\lambda\phi^4_4$ on noncommutative space

$$S[\phi] = \int_{\mathbb{R}^4} dx \Big(\frac{1}{2} \phi \;\; (-\Delta + \Omega^2 (\qquad x)^2 + \mu^2 \quad) \phi + \frac{\lambda}{4} \phi \;\; \phi \;\; \phi \;\; \phi \Big)(x)$$

Regularisation of $\lambda\phi^4_4$ on noncommutative space

$$S[\phi]=\int_{\mathbb{R}^4} dx\Big(\frac{1}{2}\phi\star(-\Delta+\Omega^2(2\Theta^{-1}x)^2+\mu^2\quad)\phi+\frac{\lambda}{4}\phi\star\phi\star\phi\star\phi\Big)(x)$$

with Moyal product $(f\star g)(x)=\int_{\mathbb{R}^4\times\mathbb{R}^4}\frac{dy\,dk}{(2\pi)^4}f(x+\tfrac{1}{2}\Theta k)\,g(x+y)\,e^{\mathrm{i}\langle k,y\rangle}$

Regularisation of $\lambda\phi^4_4$ on noncommutative space

$$S[\phi]=\int_{\mathbb{R}^4} dx \Big(\frac{Z_\Lambda}{2}\phi\star\big(-\Delta+\Omega^2(2\Theta^{-1}x)^2+\mu^2_{bare}\big)\phi+\frac{\lambda Z_\Lambda^2}{4}\phi\star\phi\star\phi\star\phi\Big)(x)$$

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matrix basis $f_{\underline{mn}}(x)=f_{m_1n_1}(x^0,x^1)f_{m_2n_2}(x^3,x^4)$

$$f_{mn}(y^0,y^1)=2(-1)^m\sqrt{\frac{m!}{n!}}\Big(\sqrt{\tfrac{2}{\theta}}y\Big)^{n-m}L_m^{n-m}\Big(\frac{2|y|^2}{\theta}\Big)e^{-\frac{|y|^2}{\theta}}$$

due to $f_{\underline{mn}}\star f_{\underline{kl}}=\delta_{\underline{nk}}f_{\underline{ml}}$ and $\int dx\, f_{\underline{mn}}(x)=V\delta_{\underline{mn}}$

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takes at $\Omega=1$ in matrix basis $f_{\underline{mn}}(x)=f_{m_1n_1}(x^0,x^1)f_{m_2n_2}(x^3,x^4)$

$$f_{mn}(y^0,y^1)=2(-1)^m\sqrt{\frac{m!}{n!}}\Big(\sqrt{\frac{2}{\theta}}y\Big)^{n-m}L_m^{n-m}\Big(\frac{2|y|^2}{\theta}\Big)e^{-\frac{|y|^2}{\theta}}$$

due to $f_{\underline{mn}}\star f_{\underline{kl}}=\delta_{\underline{nk}}f_{\underline{ml}}$ and $\int dx\,f_{\underline{mn}}(x)=V\delta_{\underline{mn}}$ the form

$$S[\Phi]=V\Big(\sum_{\underline{m},\underline{n}\in\mathbb{N}^2_{\mathcal{N}}}H_{\underline{m}}\Phi_{\underline{mn}}\Phi_{\underline{nm}}+\frac{Z^2_\Lambda\lambda}{4}\sum_{\underline{m},\underline{n},\underline{k},\underline{l}\in\mathbb{N}^2_{\mathcal{N}}}\Phi_{\underline{mn}}\Phi_{\underline{nk}}\Phi_{\underline{kl}}\Phi_{\underline{lm}}\Big)$$

$$H_{\underline{m}}=Z_\Lambda\Big(\frac{|\underline{m}|}{\sqrt{V}}+\frac{\mu^2_{bare}}{2}\Big)\,,\qquad |\underline{m}|:=m_1+m_2\le\mathcal{N}$$

- $V=\left(\frac{\theta}{4}\right)^2$ is for $\Omega=1$ the volume of the nc manifold.

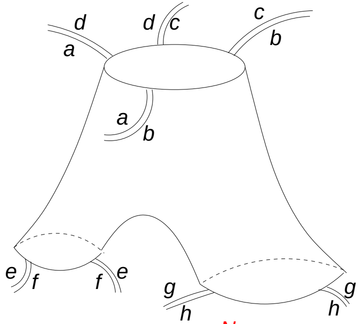
More generally: field-theoretical matrix models

Euclidean quantum field theory

- action $S[\Phi] = V\,\mathrm{tr}(E\Phi^2 + P[\Phi])$
  for unbounded positive selfadjoint operator E with compact resolvent, and $P[\Phi]$ a polynomial
- partition function $\mathcal{Z}[J] = \int \mathcal{D}[\Phi]\ \exp(-S[\Phi] + V\,\mathrm{tr}(\Phi J))$

- For $P[\Phi] = \frac{i}{6}\Phi^3$ this is the Kontsevich model which computes the intersection theory on the moduli space of complex curves. We choose $P[\Phi] = \frac{\lambda}{4}\Phi^4$.
- Perturbative expansion $e^{-V\,\mathrm{tr}(P[\Phi])} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(V\,\mathrm{tr}(P[\Phi]))^n$ leads to ribbon graphs. They encode genus-g Riemann surface with B boundary components.
- We avoid the expansion, but keep the topological structure:

Topological expansion



- Choosing $H = \mathrm{diag}(H_a)$, matrix index conserved along every strand.
- The k^{th} boundary component carries a cycle $J^{N_k}_{p_1 \ldots p_{N_k}} := \prod_{j=1}^{N_k} J_{p_j p_{j+1}}$ of N_k external sources, $N_k + 1 \equiv 1$.
- Expand $\log \mathcal{Z}[J] = \sum \frac{1}{S} V^{2-B} G_{|p^1_1 \ldots p^1_{N_1}|\ldots|p^B_1 \ldots p^B_{N_B}|} \prod_{\beta=1}^{B} J^{N_\beta}_{p^\beta_1 \ldots p^\beta_{N_\beta}}$ according to the cycle structure.
- QFT of matrix models determines the weights of Riemann surfaces with decorated boundary components compatible with
  1. gluing (of fringes)
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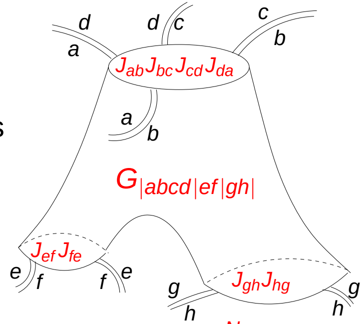

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Ward identity

Matrix Base $\quad H_{nm} = Z_\Lambda\left(\frac{(n+m)}{V} + \frac{\mu^2_{bare}}{2}\right)$

$S[\Phi] = \sum_{n,m} \frac{1}{2}\Phi_{nm}H_{nm}\Phi_{mn} + \frac{\lambda}{4}\sum_{nmpq} Z_\Lambda^2 \Phi_{nm}\Phi_{mp}\Phi_{pq}\Phi_{qn}$

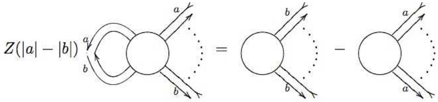
inner automorphism $\phi \mapsto U^*\Phi U$ of M_Λ, infinitesimally,

not a symmetry of the action, but invariance of measure

Interpretation

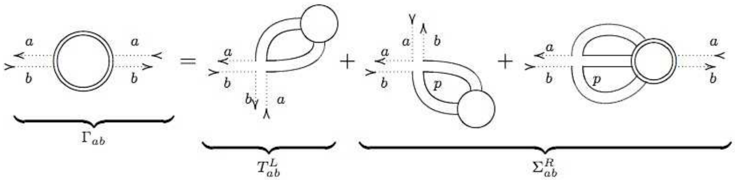
Insertion of special vertex $V^{ins}_{ab} := \sum_n (H_{an} - H_{nb})\phi_{bn}\phi_{na}$

into external face equals the difference between the exchanges of external sources $J_{nb} \mapsto J_{na}$ and $J_{an} \mapsto J_{bn}$

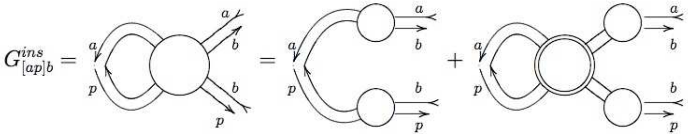


The dots stand for the remaining face indices.

SD equation 2



- vertex is $Z_\Lambda^2\lambda$, connected two-point function is G_{ab}: first graph equals $Z_\Lambda^2\lambda\sum_q G_{aq}$
- open p-face in Σ^R and compare with insertion into connected two-point function



gives for 2 point function:

$$Z_\Lambda^2\lambda\sum_q G_{aq} - Z_\Lambda\lambda\sum_p (G_{ab})^{-1}\frac{G_{bp}-G_{ba}}{|p|-|a|} = H_{ab} - G_{ab}^{-1}\,.$$

Schwinger-Dyson equations (for $S_{int}[\Phi] = \frac{\lambda}{4}\mathrm{tr}(\Phi^4)$)

In a scaling limit $V \to \infty$ and $\frac{1}{\sqrt{V}} \sum_{p\in I}$ finite, we have:

1. A closed non-linear equation for $G_{|ab|}$

$$G_{|ab|} = \frac{1}{E_a + E_b} - \frac{\lambda}{(E_a+E_b)}\frac{1}{V}\sum_{p\in I}\Big(G_{|ab|}G_{|ap|} - \frac{G_{|pb|} - G_{|ab|}}{E_p - E_a}\Big)$$

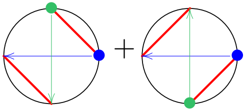
2. For $N \geq 4$ a universal algebraic recursion formula

$$G_{|b_0 b_1 \dots b_{N-1}|} = (-\lambda)\sum_{l=1}^{\frac{N-2}{2}} \frac{G_{|b_0 b_1 \dots b_{2l-1}|}G_{|b_{2l} b_{2l+1} \dots b_{N-1}|} - G_{|b_{2l} b_1 \dots b_{2l-1}|}G_{|b_0 b_{2l+1} \dots b_{N-1}|}}{(E_{b_0} - E_{b_{2l}})(E_{b_1} - E_{b_{N-1}})}$$

- scaling limit corresponds to restriction to genus $g = 0$
- similar formulae for $B \geq 2$
- no index summation in $G_{|abcd|}$ $\Rightarrow$ β-function zero!

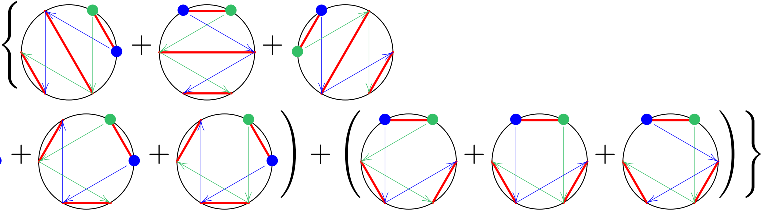
Graphical realisation

$$G_{b_0b_1b_2b_3} = (-\lambda)\frac{G_{b_0b_1}G_{b_2b_3} - G_{b_0b_3}G_{b_2b_1}}{(b_0-b_2)(b_1-b_3)} = -\lambda\Big\{$$



$$\Big\}$$

$$G_{b_0\dots b_5} = \lambda^2\Big\{$$



$$\Big\}$$

b_i b_j $= G_{b_ib_j}$ leads to non-crossing chord diagrams; these are counted by the Catalan number $C_{\frac{N}{2}} = \frac{N!}{(\frac{N}{2}+1)!\frac{N}{2}!}$

b_i  b_j $= \frac{1}{b_i - b_j}$ leads to rooted trees connecting the even or odd vertices, intersecting the chords only at vertices

Back to $\lambda\Phi^4_4$ on Moyal space

- Infinite volume limit (i.e. $\theta \to \infty$) turns discrete matrix indices into continuous variables $a, b, \dots \in \mathbb{R}_+$ and sums into integrals
- Need energy cutoff $a, b, \dots \in [0, \Lambda^2]$ and normalisation of lowest Taylor terms of two-point function $G_{|nm|} \mapsto G_{ab}$
- Carleman-type singular integral equation for $G_{ab} - G_{a0}$

Theorem (2012/13) (for $\lambda < 0$, using $G_{b0} = G_{0b}$)

Let $\mathcal{H}^\Lambda_a(f) = \frac{1}{\pi}\mathcal{P}\int_0^{\Lambda^2} \frac{f(p)\,dp}{p-a}$ be the *finite Hilbert transform*.

$$G_{ab} = \frac{\sin(\tau_b(a))}{|\lambda|\pi a} e^{\mathrm{sign}(\lambda)(\mathcal{H}^\Lambda_0[\tau_0(\bullet)] - \mathcal{H}^\Lambda_a[\tau_b(\bullet)])}$$

where $\tau_b(a) := \arctan_{[0,\pi]}\left(\frac{|\lambda|\pi a}{b + \frac{1+\lambda\pi a \mathcal{H}^\Lambda_a[G_{\bullet 0}]}{G_{a0}}}\right)$ and G_{a0} solution of

$$G_{b0} = G_{0b} = \frac{1}{1+b}\exp\left(-\lambda\int_0^b dt \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1+\lambda\pi p \mathcal{H}^\Lambda_p[G_{\bullet 0}]}{G_{p0}}\right)^2}\right)$$

Discussion

Together with explicit (but complicated for $G_{ab|cd}, G_{ab|cd|ef}, \dots$) formulae for higher correlation functions, we have exact solution of $\lambda\phi^4_4$ on extreme Moyal space in terms of

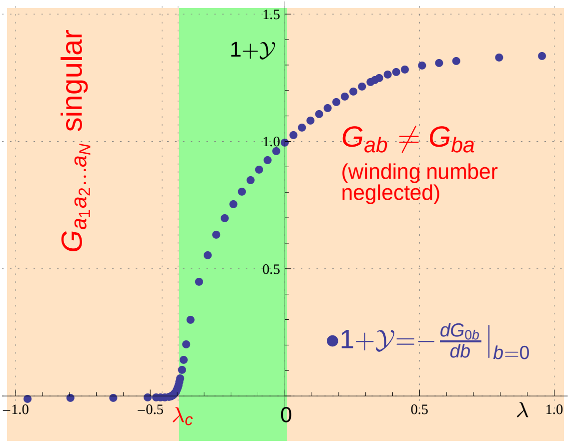
$$G_{b0} = G_{0b} = \frac{1}{1+b} \exp\left(-\lambda \int_0^b dt \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1+\lambda\pi p \mathcal{H}^\Lambda_p[G_{\bullet 0}]}{G_{p0}}\right)^2} \right)$$

Possible treatments

1. perturbative solution: reproduces all Feynman graphs, generates polylogarithms and ζ-functions
2. iterative solution on computer: nicely convergent, find interesting phase structure
3. rigorous existence proof of a solution
4. work in progress: (guess); should give uniqueness

Computer simulation: evidence for phase transitions

piecewise linear approximation of G_{0b}, G_{ab} for $\Lambda^2 = 10^7$ and 2000 sample points. Consider $1+\mathcal{Y} := -\frac{dG_{0b}}{db}\Big|_{b=0}$



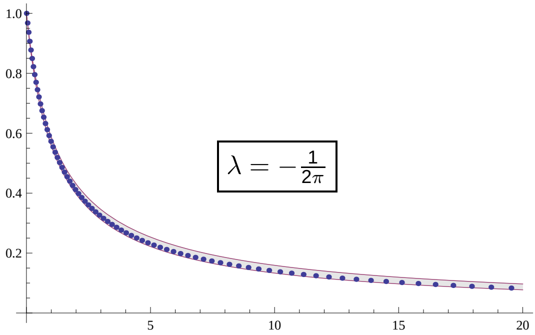
- $(1+\mathcal{Y})'(\lambda)$ discontinuous at $\lambda_c = -0.39$
- order parameter $b_\lambda = \sup\{b \,:\, G_{0b} = 1\}$ non-zero for $\lambda < \lambda_c$
- A key property for Schwinger functions is realised in $]\lambda_c, 0]$, outside?

Fixed point theorem

Theorem (2015)

Let $-\frac{1}{6} \leq \lambda \leq 0$. Then the equation has a C_0^1-solution

$$\frac{1}{(1+b)^{1-|\lambda|}} \leq G_{0b} \leq \frac{1}{(1+b)^{1-\frac{|\lambda|}{1-2|\lambda|}}}$$



Proof via Schauder fixed point theorem.

This involves continuity and compactness of a certain operator (in norm topology)

An analogy

2D Ising model	4D nc ϕ^4-theory
temperature T, $K = \frac{J}{k_B T}$	frequency Ω
Kramers-Wannier duality $\sinh(2K)\sinh(2K^*) = 1$	Langmann-Szabo duality $\Omega\Omega^* = 1$
solvable at $K = K^*$ scale-invariant	solvable at $\Omega = \Omega^*$ almost scale-invariant
CFT minimal model ($m = 3$)	matrix model
operator product expansion Virasoro constraints	Schwinger-Dyson equation Ward identities
critical exponents $G^{\sigma\sigma}_{n0} \propto \frac{1}{n^{d-2+\eta}}$, $\eta = \frac{1}{4}$	critical exponents $G^{\phi\phi}_{n0} \propto \frac{1}{n^{2+\eta}}$, $\lambda \in]\lambda_c, 0]$
Virasoro algebra, CFT, subfactors, ...	**???**

Relativistic and Euclidean quantum field theory

- We define a QFT by Wightman distributions
  $\mathcal{W}_N(x_1, \dots, x_N) = W_N(x_1 - x_2, \dots, x_{N-1} - x_N)$
- Theorem: The W_N are boundary values of holomorphic functions (on permuted extended forward tube $\subset \mathbb{C}^{4(N-1)}$)
  Euclidean points (minus diagonals) defines Schwinger functions
- Schwinger functions inherit properties such as real analyticity, Euclidean invariance and complete symmetry
- Hence, moments of probability distributions provide candidate Schwinger functions (link to statistical physics)

Theorem (Osterwalder-Schrader, 1974)

One additional requirement, reflection positivity, leads back to Wightman theory

From matrix model to Schwinger functions on $\mathbb{R}^4$

reverting harmonic oscillator basis, $1+\mathcal{Y}:=-\frac{dG_{0b}}{db}\Big|_{b=0}\dots$

Theorem (2013): *connected* Schwinger functions

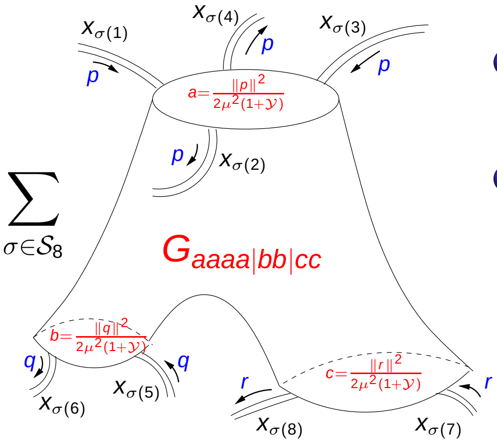
$$S_c(\mu x_1,\dots,\mu x_N) = \frac{1}{64\pi^2}\sum_{\substack{N_1+\dots+N_B=N\\ N_\beta \text{ even}}}\sum_{\sigma\in\mathcal{S}_N}\Bigg(\prod_{\beta=1}^{B}\frac{4^{N_\beta}}{N_\beta}\int_{\mathbb{R}^4}\frac{d^4p_\beta}{4\pi^2\mu^4}\, e^{\mathrm{i}\left\langle \frac{p_\beta}{\mu},\sum_{i=1}^{N_\beta}(-1)^{i-1}\mu x_{\sigma(N_1+\dots+N_{\beta-1}+i)}\right\rangle}\Bigg)$$

$$\times G_{\underbrace{\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})},\dots,\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_1}\Big|\dots\Big|\underbrace{\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})},\dots,\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_B}}$$

Hidden noncommutativity: have internal interaction of matrices; commutative subsector propagates to outside world

- Schwinger functions are symmetric and invariant under full Euclidean group (completely unexpected for NCQFT!)
- remains: reflection positivity
- finally: Is it non-trivial?

Connected (4+2+2)-point function



1. individual Euclidean symmetry in every boundary component (no clustering)
2. particle scattering without momentum exchange
   - in 4D a sign of triviality (mind assumptions!)
   - familiar in 2D models with factorising S-matrix
   - a consequence of integrability

Is there a precise link between exact solution of our 4D model and traditional integrability known from 2D?

Osterwalder-Schrader reflection positivity

Proposition (2013)

$S(x_1, x_2)$ is reflection positive iff $a \mapsto G_{aa}$ is a Stieltjes function,

$$G_{aa} = \int_0^\infty \frac{d(\rho(t))}{a+t}, \qquad \rho \text{ – positive measure.}$$

Excluded for any $\lambda > 0$ (unless rescued by winding number)

- naïve anomalous dimension η positive for $\lambda > 0$,
- renormalisation oversubtracts: η_{ren}, λ of opposite sign
- p-space 2-point function $\frac{1}{(p^2+m^2)^{1-\eta/2}}$
  Positivity and convergence contradict each other!
- Need (analytical?) continuation between
  – one regime where existence can be proved and
  – another regime where positivity holds.

Reflection positivity simplifies the problem

If G_{x0} is Stieltjes, then Hilbert transform can be avoided:

$$\frac{G_{xy}}{G_{x0}} = \exp\left(-\frac{1}{\pi}\int_1^\infty \frac{dt}{t+x} \underset{[0,\pi]}{\arctan}\left(\frac{y\,\mathrm{Im}(G_{-(t+\mathrm{i}\epsilon),0})}{1-\lambda t\int_0^\infty ds\,\frac{G_{s0}}{t+s}+y\,\mathrm{Re}(G_{-(t+\mathrm{i}\epsilon),0})}\right)\right)$$

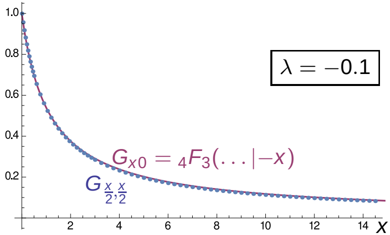
Which class of functions has desired analyticity+holomorphicity and manageable integral transforms?

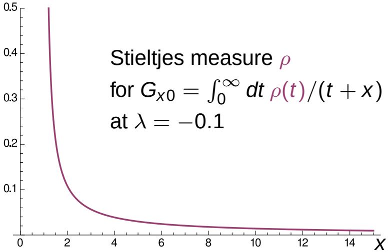
hypergeometric functions $G_{x0} = {}_nF_{n-1}\left(\begin{matrix}a, b_1, \dots b_{n-1}\\ c_1, \dots, c_{n-1}\end{matrix}\middle| -x\right)$ if $a \in [0,1]$ and $c_i > b_i > a$

- holomorphicity at $y > 0$: determine a, b_i, c_i by $G^{(k)}_{0y} = G^{(k)}_{y0}$
- find: $a = 1 + \frac{1}{\pi}\arcsin(\lambda\pi)$, $\prod_{i=1}^n \frac{c_i - 1}{b_i - 1} = \frac{\arcsin(\lambda\pi)}{\lambda\pi}$
- critical coupling constant is $\lambda_c = -\frac{1}{\pi} = -0.3183\dots$

Källén-Lehmann spectrum

- Numerics makes it completely clear (but doesn't prove) that G_{x0} is Stieltjes
- reflection positivity equivalent to G_{xx} a Stieltjes function
- the shape makes this plausible:





- measure for G_{x0} has mass gap $[0,1[$, but no further gap (remnant of UV/IR-mixing)
- absence of the second gap (usually $]1,4[$) circumvents triviality theorems

Summary

1. $\lambda\phi^4_4$ on nc Moyal space is, at infinite noncommutativity, exactly solvable in terms of a fixed point problem
   - theory defined by quantum equations of motion (= Schwinger-Dyson equations)
   - existence proved for $-\frac{1}{6}<\lambda\leq 0$
   - phase transitions and critical phenomena
2. Projection to Schwinger functions for scalar field on $\mathbb{R}^4$ = hidden noncommutativity
   - full Euclidean symmetry (completely unexpected)
   - no momentum exchange (close to triviality), possibly a consequence of integrability
   - numerical approach with tiny error: leaves no doubt that Schwinger 2-point function is reflection positive for $-\frac{1}{\pi}<\lambda\leq 0$

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3. ready to embark on higher Schwinger functions