The generalised principle of perturbative agreement and the thermal mass

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Quantum Field Theory: Infrared problems and constructive aspects,
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Motivations

- **Quantum filed theory** - Incredible accordance with experiments.
- Rigorous theory only in the **linear case**

  (or for some low dim. theories or some integrable models)
- Interacting theories treated by **perturbation theory**.

  Problems:
  - **Ultraviolet divergences: Solved** by local perturbation and renormalization.
  - **Infrared problems:** Local measurements, adiabatic limit.
  - Solutions are given as a **formal power series** in the coupling constants. (The series is at most an **asymptotic expansion**)

- Nowadays **perturbative** methods are well understood also on **curved spacetimes**

Motivations

In any interacting theory treated perturbatively there is an **ambiguity**:

$$\mathcal{S} = (\mathcal{S}_1 + Q) + V = \mathcal{S}_1 + (Q + V)$$

$\mathcal{S}_1$ free action, Q quadratic perturbation, V generic perturbation.

We might consider either V or $Q + V$ as perturbation potential.

- The two perturbative constructions must agree

Principle of perturbative agreement

We might use this freedom:

- Move a quadratic part of the potential to the free one.
- Perturbative constructions around massless theories tend to have severe infrared divergences.

Plan of the talk

- Perturbative algebraic quantum field theory.
- The principle of perturbative agreement and its generalization.
- An application: The thermal mass.

This talk is based on

- N. Drago, T.-P. Hack, NP, [arXiv:1502.02705].

Bibliography

- S. Hollands, R. Wald "Interacting AQFT on curved spacetimes"
- R. Brunetti, M. Duetsch, K. Fredenhagen, K. Rejzner "pAQFT"
- K. Fredenhagen, F. Lindner CMP 332, 895 (2014)

Perturbative algebraic quantum field theory.

AQFT in the local and covariant formulation

- On a globally hyperbolic spacetime (M, g), consider a theory described by an action of the form

$$\mathcal{S}(\phi) = \frac{1}{2} \int_M \left(\nabla^\mu \phi \nabla_\mu \phi + m^2 \phi^2 + V(\phi) \right) d\mu_g \, .$$

- **Quantize** that system on a **curved background**:
  - No notion of energy, no notion of vacuum, no symmetry, many inequivalent representations are possible.

- **Algebraic approach**: divide the problem in two parts:
  - Identify the observable and their algebraic structure $\mathscr{A}$ (implementing the commutation relations)
  - Once a state $\omega : \mathscr{A} \to \mathbb{C}$ is given the **GNS** theorem furnishes an Hilbert space representation.

Functional approach *[Brunetti Duetsch Fredenhagen]*

- (Off-shell) **Field Configurations**: $\phi \in \mathcal{E}(M;\mathbb{R})$
- **Observables**: functionals over (off-shell) field configurations

$$\mathscr{F} := \{F : \mathcal{E}(M) \to \mathbb{C},\ \text{smooth}\ , F^{(n)} \in \mathcal{E}'(M), ...\}$$

  - $\mathscr{F}_{\mathrm{reg}}$ **Regular functionals**: $F^{(n)} \in \mathcal{D}(M^n)$, example

$$F_f := \int_M \phi(x)\, f(x)\, d\mu_g .$$

  - $\mathscr{F}_{\mu c}$ **Microcausal functionals**:

$$\mathscr{F}_{\mu c} = \left\{ F \in \mathscr{F} \,\middle|\, \mathrm{WF}(F^{(n)}) \cap (\overline{V}_+^n \cup \overline{V}_-^n) = \emptyset \right\}$$

  - $\mathscr{F}_{\mathrm{loc}}$ **Local functionals**: contains fields of the form $\int_M \phi(x)^n\, f(x)\, d\mu_g$
- To get the $*-$**Algebra of field observables** we need a product indicated by $\star$.

$$\mathscr{A} = (\mathscr{F}, \star, *)$$

- The product needs to be compatible with **local covariance**.

Linear theories

- In a free (linear) theory

$$P\phi = -\Box\phi + m^2\phi = 0$$

- the product can be written explicitly by a **contraction exponential**

$$F \star G(\phi) = F(\phi)G(\phi) + \sum_{n\geq 1} \frac{\hbar^n}{n!} \left\langle \Delta^{\otimes n}, F^{(n)}(\phi) \otimes G^{(n)}(\phi) \right\rangle$$

where $\Delta = \Delta^R - \Delta^A$ is the **causal propagator**,

- the resulting $\star$ is well defined on $\mathscr{F}_{\mathrm{reg}}$
  - It implements the **canonical commutation relations**
  - It is a formal deformation of the pointwise product, $\Longrightarrow$ **deformation quantization**.

Extension to encompass local fields

- Local non linear fields (like $\int_M \phi^n f \, d\mu_g$) can be added after deforming the product: using Δ^+ (a generic **Hadamard function**) instead of Δ
  - Δ^+ characterized by **microlocal spectrum condition** *[Radzikowski]*

$$\mathrm{WF}(\Delta^+) = \left\{(x,x';k,k') \in T^*M^2 \setminus \{0\} \,|(x,k) \sim (x',-k'), k \triangleright 0\right\}$$

  - Δ^+ is non-unique, different propagators produces isomorphic algebras

$$\Delta^+(x,y) := \frac{1}{8\pi^2}\left(\frac{u(x,y)}{\sigma_+(x,y)} + v(x,y)\log(\lambda^{-2}\sigma_+(x,y)) + w(x,y)\right)$$

$$\sigma_+(x,y) := \sigma(x,y) + i\epsilon(t_x - t_y) + \epsilon^2$$

  - Its antisymmetric part coincides with Δ
  - The new product $\star_{\Delta^+}$ can be extended to **microcausal functionals**

$$\mathscr{F}_{\mu c} = \left\{F \in \mathscr{F} \,\middle|\, \mathrm{WF}(F^{(n)}) \cap (\overline{V}_+^n \cup \overline{V}_-^n) = \emptyset\right\}$$

  - It is the algebra generated by Wick products of fields

Time ordered products

- To construct an interacting theory by means of perturbation theory we need to introduce the **time ordered product** $\cdot_T$.

- On regular functionals $\cdot_T$ is the symmetric product characterized by the **causal factorization** property:

$$F \cdot_T G = F \star G \qquad \text{if } F \gtrsim G,$$

- It is a contraction exponential constructed with the Feynman propagator Δ^F in the place of Δ^+.

$$\Delta^F(x,y) := \frac{1}{8\pi^2}\left(\frac{u(x,y)}{\sigma_F(x,y)} + v(x,y)\log(\lambda^{-2}\sigma_F(x,y)) + w(x,y)\right)$$

$$\sigma_F(x,y) := \sigma(x,y) + i\epsilon$$

Perturbative constructions of interacting theories

- Interacting observables are represented over $\mathscr{A}$ by means of the **Bogoliubov formula** which gives rise to the **Møller operator**

$$\mathscr{R}_V^{\hbar}(F) = \frac{d}{d\lambda} S(V)^{-1} \star S(V + \lambda F)\bigg|_{\lambda=0} = S(V)^{-1} \star (S(V) \cdot_T F),$$

- Where the $S-$matrix is the **time ordered exponential** of V.

$$S(V) = \exp_T\left(\frac{i}{\hbar} V\right) = \sum_{n \geq 0} \frac{i^n}{n! \hbar^n} \underbrace{V \cdot_T \ldots \cdot_T V}_{n \text{ times}},$$

- Used to represent the interacting algebra over the free one.

Bogoliubov formula and interacting algebras

- Linear fields are weak solutions of the equation of motion

$$\mathscr{R}^{\hbar}_V(F_{Pf}) = F_{Pf} + \mathscr{R}^{\hbar}_V(\langle V^{(1)}, f\rangle)$$

$$Pf = -\Box f + m^2 f$$

- If $\mathscr{R}^{\hbar}_V$ can be inverted we obtain the interacting $\star-$product

$$F \star_V G = (\mathscr{R}^{\hbar}_V)^{-1}\left(\mathscr{R}^{\hbar}_V(F) \star \mathscr{R}^{\hbar}_V(G)\right)$$

  and thus construct

$$\widetilde{\mathscr{A}}_V = (\mathscr{F}, \star_V, *_V)$$

- For a generic local V the construction of $\cdot_T$ is more subtle.

Time ordered maps

- The extension to local functionals is not trivial:
  - $(P\Delta^F) = i\delta \mod C^\infty$ hence ${\Delta^F}^n$ is not well defined.
  - ${\Delta^F}^n$ is well defined outside the diagonal $M^2 \setminus D$.

- The time ordered product among local functionals is constructed introducing a time ordered map $T : \mathscr{F}_{\mathrm{loc}}^{\otimes n} \to \mathscr{F}$
  - To construct T we need to employ **renormalization**. top
  - T is not unique.
  - $T : \mathscr{F}_{\mathrm{loc}} \to \mathscr{F}_{\mathrm{loc}}$ fixes the renormalization freedom.
  - $F \cdot_T G = T(T^{-1}(F), T^{-1}(G))$.
  - Satisfying a set of good axioms *[Hollands, Wald]*:
    Causal factorization, covariance, ...

- By **covariance**, it has to be understood as $T(g, m^2, \star)$ namely the evaluation of a map on the particular spacetime.

Algebraic adiabatic limit *[Brunetti Fredenhagen 2005]*

- Up to now we have considered interacting potential V_g with compact support

$$V_g = \int g \mathcal{L}_I d\mu$$

where $\mathcal{L}_I$ is the interacting Lagrangean density and g is a compactly supported smooth function.

- To describe a physical theory we want to analyze the limit $g \to 1$.

- The **causal factorization property** for the $S-$matrix

$$S(A+B+C) = S(A+B) \star S(B)^{-1} \star S(B+C)$$

if $J^+(\mathrm{supp} A) \cap J^-(\mathrm{supp}\, C) = \emptyset$.

- From the **causal factorization property** it follows that:
  - $\mathscr{R}^{\hbar}_{Vg}(F)$ depends only on $\mathcal{L}_I$ "restricted" to $J^-(\mathrm{supp}F)$.
  - Consider a double cone O and a functional F such that $\mathrm{supp}F \subset O$. Fix $g = 1$ and $g' = 1$ on O. It is possible to find an unitary operator $U_{g,g'}$ such that

$$\mathscr{R}^{\hbar}_{g'V}(F) = U^{-1}_{g,g'} \star \mathscr{R}^{\hbar}_{gV}(F) \star U_{g,g'}$$

    for every F with $\mathrm{supp}F \subset O$.
  - Hence $\mathscr{A}_{Vg}(O)$ are **unitarily equivalent** for every $g = 1$ on O.
  - We can define the bundle

$$\bigcup_{g=1 \text{ on } O} \{g\} \times \mathscr{A}_g(O)$$

  - Then $\mathscr{A}_{V1}(O)$ is the set of **"constant sections"** (whose components are connected by some unitary transformation).
- This is the **algebraic adiabatic limit**.
- The analysis of what happens to states is still difficult.

The principle of perturbative agreement

The principle of perturbative agreement

- Quadratic potential over a linear theory

$$\mathcal{S}_1 + Q = \mathcal{S}_2, \qquad Q = \frac{1}{2}\int_M \delta m^2 \phi^2 d\mu_g$$

  Q with compact support
- We have $\mathscr{A}_1$, $\mathscr{A}_2$ and $\widetilde{\mathscr{A}_{1,Q}}$

Question

What is the relation between $\mathscr{A}_2$ and $\widetilde{\mathscr{A}_{1,Q}}$?

Classical Møller operator

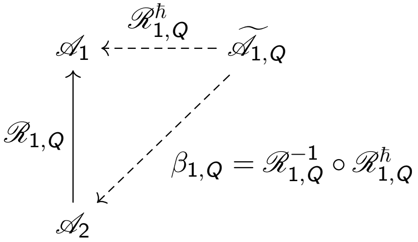
- $\widetilde{\mathscr{A}}_{1,Q}$ is imbedded into $\mathscr{A}_1$ by the quantum Møller operator $\mathscr{R}^{\hbar}_{1,Q}$.
- We need to imbed $\mathscr{A}_2$ into $\mathscr{A}_1$.
  - Identify the algebras in the complement of $J^+(\mathrm{supp}Q)$.
  - Using linear dynamics (time-slice axioms).

- Formally the operator which realizes this imbedding is called **classical Møller operator** and it is indicated by

$$\mathscr{R}_{1,Q} : \mathscr{A}_2 \to \mathscr{A}_1, \qquad \mathscr{R}_{1,Q}(F(\phi)) = F(R_Q(\phi))$$

where $R_Q = \mathbb{I} - \Delta_2^R \circ Q^{(1)}$ intertwines between the two dynamics

$$(-\Box + m^2 + \delta m^2) R_Q(\phi) = (-\Box + m^2)(\phi)$$



Proposition

- $\mathscr{R}_{1,Q}$ *is a* $*-$**isomorphism**
- *On* $\mathscr{F}_{\mathrm{reg}}$ $\mathscr{R}^{\hbar}_{1,Q}$ *can be inverted, hence* $\widetilde{\mathscr{A}}^{\mathrm{reg}}_{1,Q}$ *is well defined*
- $\widetilde{\mathscr{A}}^{\mathrm{reg}}_{1,Q}$ *is* $*-$**isomorphic** *to* $\mathscr{A}^{\mathrm{reg}}_2$
- *The isomorphism is realized by* $\beta_{1,Q} = \mathscr{R}^{-1}_{1,Q} \circ \mathscr{R}^{\hbar}_{1,Q}$
- $\beta_{1,Q}$ *is a* **non trivial deformation** *of* $\mathscr{A}^{\mathrm{reg}}_2$

$$\beta_{1,Q} = \alpha_d, \qquad \alpha_d(F) = \sum \frac{\hbar}{n!}\langle d^{\otimes n}, F^{(2n)}\rangle, \qquad d = \Delta^F_2 - \Delta^F_1$$

The principle of perturbative agreement for mass pert.

- Extension to **more general functionals** needs to be done perturbatively because $(\Delta_2^F - \Delta_1^F)^2$ is not well defined.
- Logarithmic divergences pop up. Renormalization needs to be employed.
- It can be chosen in such a way that the **PPA** of Hollands and Wald can be satisfied:

[Hollands Wald 2005] The time ordered map T, considered as a map $T(g, M, \star)$, is said to satisfy the **Principle of Perturbative Agreement (PPA)** if,

$$T(g, M_2, \star_2) = \beta_{1,Q} \circ T(g, M_1, \star_1).$$

on ${\mathscr{F}_{\mathrm{loc}}}^{\otimes n}$.

Proposition

- $T_2 = \beta_{1,Q} \circ T_1$ *defines a* **time ordered map** *for* $\mathscr{A}_2$ *for every* T_1 *on* $\mathscr{A}_1$*.*
- $\beta_{1,Q}$ *satisfies a* **cocycle condition**

$$\beta_{1,Q_3} = \beta_{2,Q_3} \circ \beta_{1,Q_2}$$

- Hence to get a map T which satisfies the perturbative agreement in the sense of Hollands and Wald for mass variations we operate as follows
  - Fix $T = T_1(M_1 = 0, g)$
  - Consider $T(M_2 = M, g) = \beta_{1,Q} T(0, g)$

Generalization to interacting theories

Proposition

The map $\beta_{1,Q}$ *to connects* $\mathcal{S}_1 + (Q + V)$ *to* $\mathcal{S}_2 + V$

$$\mathscr{R}^{\hbar}_{1,Q+V} = \mathscr{R}_{1,Q} \circ \mathscr{R}^{\hbar}_{2,V} \circ \beta_{1,Q} \,.$$

KMS states over massless solutions

An application: the thermal mass

- Perturbative construction of a **thermal (KMS) state** for a massless $\lambda\phi^4$ theory in Minkowski spacetime.

- For massive theories this is done by *[Fredenhagen and Lindner 2014]*, using ideas coming from **statistical mechanics** and translating them in the language of pertrubative quantum field theory:

$$\omega^I_\beta(A) = \frac{\omega_\beta(AU(i\beta))}{\omega_\beta(U(i\beta))}$$

  - ω_β the KMS state at inverse temperature β w.r.t. to a Minkowskian time for the free theory.
  - U is the **intertwiner** of α^I_t and α_t the interacting and free dynamics:
- In ordinary statistical mechanics $U(t) = e^{it(H_0+H_I)}e^{-itH_0}$
- In QFT it is not possible to construct $\exp iH_I(t)t$ at fixed time.
- Circumvent the problem analyzing directly the intertwiner.

- Restrict the algebra in a **timeslab** $\mathcal{T}$. (Done by the **time slice axiom**).
- Restrict the interaction in a neighborhood of $\mathcal{T}$. (With a **time cutoff** χ)
- Consider a **space compact interaction** (inserting a **spatial cutoff** h).

$$V = \int h\chi\mathcal{H}_I d\mu$$

(the adiabatic limit correspond to the removal of the spatial cutoff)
- With χ and h, the **causal factorization property** of S permits
  - to construct $U(t)$

$$\alpha_t^I(A) = U(t)\alpha_t(A)U(t)^{-1}$$

  - to show the cocycle relation

$$U(t+s) = U(t)\alpha_t(U(s))$$

  - to write its generator.
  - to show that $U(t)$ state does not depend on χ.
- Construct the state $\omega^I_{\beta,h}$ with $U(t)$. $= \omega_\beta(AU(i\beta))/\omega_\beta(U(i\beta))$
- The **thermal correlation functions** decay exponentially for large spatial separations.
- We can take the **adiabatic limit** $h \to 1$.

Generalization to massless interacting theories

- The difficulty in the construction of this state over massless theories is in the **adiabatic limit**.
- For perturbations over **massless** theories the decay of thermal correlations does not permit to take the adiabatic limit.
- To obtain the interacting KMS state over a massless theory we proceed as follows:
  - When represented in $(\mathscr{A}, \star_{\omega_\beta})$ ϕ^4 acquires effectively a mass: $\phi^4 + M_\beta^2\phi^2 + C$.
  - Move this mass to the free theory by means of the map $\beta_{1,Q}$ constructed above. (Generalized principle of perturbative agreement)
  - Construct there the KMS state for the interacting theory as shown before.

Summary

- Perturbative algebraic quantum field theory.
- Mass perturbations can be treated in two ways. The resulting algebras are $*-$homeomorphic. The homomorphism is not trivial.
- It can be used to add “virtual masses” to interacting field theories.
- KMS states for massless interacting theories can be constructed also in the adiabatic limit.

Thanks a lot for your attention!

Expansion in terms of Feynman graphs

$$V_1 \cdot_T \cdots \cdot_T V_n = T_n(V_1 \otimes \cdots \otimes V_n)$$

$$T_n = e^{\sum_{1\le i<j\le n} \left\langle \Delta^F, \frac{\delta^2}{\delta\phi_i \delta\phi_j} \right\rangle} = \sum_{\Gamma \in \mathcal{G}_n} \frac{1}{N(\Gamma)} \left\langle \tau_\Gamma, \frac{\delta^{2|E(\Gamma)|}}{\prod_{i\in V(\Gamma)} \prod_{E(\Gamma)\ni e \supset i} \delta\phi_i(x_i)} \right\rangle,$$

where $\mathcal{G}_n$ is the set of graphs with n vertices

$$\tau_\Gamma = \prod_{1\le i<j\le n} \Delta^F(x_i, x_j)^{l_{ij}}.$$

and l_{ij} counts the number of edges joining the vertices i, j.

Epstain Glaser recursive construction

- **Epstein and Glaser** used a recursive procedure over the number of local fields to construct the time ordered products among elements of $\mathscr{F}_{loc}$.
- At step n we need to construct all the possible **"Feynman diagrams"** with n vertices. These are distributions formed with Δ^F.
- **causal factorization** $F \cdot_T G = F \star_H G$ if $F \gtrsim G$ fixes these products "up to the total diagonal" $\mathcal{D} \subset \mathscr{M}^n$.
- Extension is in general **not unique** $\Longrightarrow$ renormalization freedom.
- Requiring that the extension preserves the **Steinmann** scaling degree the freedom is largely restricted. *[Brunetti Fredenhagen 2000, Hollands Wald 2002]* back