Algebraic adiabatic limit in theories with local symmetries

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Outline of the talk

1. Classical theory
2. Quantization
    - Non-renormalized theory
    - Renormalization
3. Adiabatic limit

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- Spacetime M: a smooth manifold with a smooth pseudo-Riemannian metric of the Lorentz signature (we choose the convention $(+,-,-,...,-)$). We also assume M to be globally hyperbolic (has a Cauchy surface).

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- Lagrangian (will be defined later).

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where φ is a field configuration, f is a density-valued function on the jet bundle over M and $j^k_x(\varphi)$ is the jet of φ at x.
- In classical theory it is enough to consider functionals that are multilocal, i.e. they are sums of products of local functionals, where $(F \cdot G)(\varphi) \doteq F(\varphi)G(\varphi)$. Let $\mathcal{F}$ denote the space of multilocal functionals on $\mathcal{E}$.

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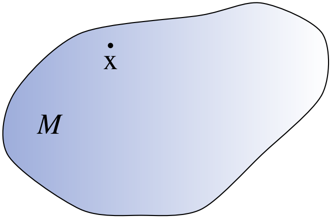

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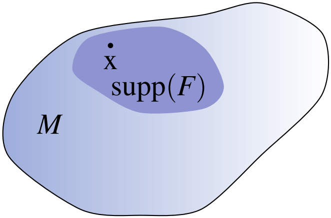

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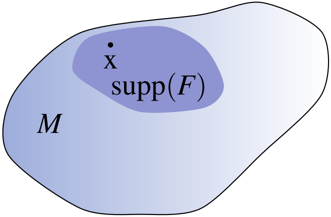

- More precisely:
$$\text{supp}\, F = \{x \in M | \forall \text{ neighbourhoods } U \text{ of } x\ \exists \varphi_1, \varphi_2 \text{ configurations}, \\ \text{supp}\, \varphi_2 \subset U \text{ such that } F(\varphi_1 + \varphi_2) \neq F(\varphi_1)\} \,.$$

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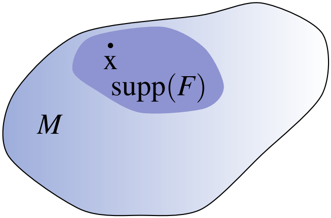

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- We assume our functionals to be compactly supported.

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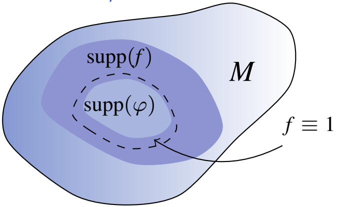
- An action S is an equivalence class of Lagrangians (i.e. $S = [L]$) under $L \sim \tilde{L}$ if $\text{supp}(L - \tilde{L})(f) \subset \text{supp}\, df$ (differ by a total divergence).

Equations of motion and symmetries

- The Euler-Lagrange derivative of $S = [L]$ is defined as $\left\langle S'(\varphi_0), h\right\rangle = \left\langle L(f)^{(1)}(\varphi_0), \varphi\right\rangle$, where $f \equiv 1$ on suppφ.



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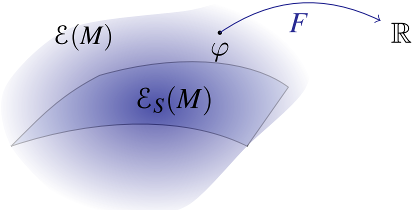

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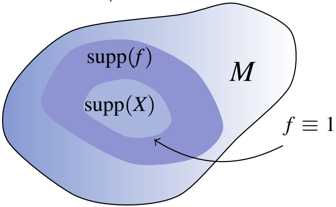

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- A local symmetry of S is a direction in $\mathcal{E}$ in which the action is constant, i.e. it is a vector field $X \in \Gamma_c(T\mathcal{E})$ such that $\partial_X S(\varphi_0) \equiv \left\langle S'(\varphi_0), X(\varphi_0) \right\rangle = 0$, $\forall \varphi_0 \in \mathcal{E}$.

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- Let $\mathcal{V}$ denote the space of multilocal vector fields on $\mathcal{E}$. We obtain a sequence:

$$0 \to Ker(\delta) \hookrightarrow \mathcal{V} \xrightarrow{\delta} \mathcal{F} \to 0,$$

  with $H_0(\delta) = \mathcal{F}_S$.

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- For the scalar field there ar no non-trivial local symmetries, so the differential graded algebra $(\Lambda\mathcal{V}, \delta)$ is a resolution of $\mathcal{F}_S$.

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- Objective: characterize $\mathcal{F}_S^{\mathrm{inv}}$, the space of gauge equivariant functionals on the space of solutions to EOM's.

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- For functions we set $\gamma F(\xi) \doteq \rho(\xi)F$, i.e. $\gamma F = \rho(c)F$, where $c^\mu(x)$ is the evaluation functional on $\mathfrak{g}$ (ghost). For forms $\gamma c = \frac{1}{2}[c, c]$. Next we extend γ by the graded Leibniz rule.

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- Note that the kernel of γ in degree 0 contains $F \in \mathcal{F}$ such that $\gamma F(\xi) = 0$ for all $\xi \in \mathfrak{g}$. Hence

$$\mathcal{F}^{\mathrm{inv}} = H^0(\mathcal{F}\widehat{\otimes}\Lambda\mathfrak{g}', \gamma).$$

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- A standard result from homological algebra allows to conclude that $H^0(\mathcal{BV}, s) = H^0(H_0(\mathcal{BV}, \delta), \gamma) = \mathcal{F}_S^{\text{inv}}$.

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- In this notation:
$$\{X, Y\} = -\int \left(\frac{\delta X}{\delta\varphi(x)} \frac{\delta Y}{\delta\varphi^\ddagger(x)} + (-1)^{|X|} \frac{\delta X}{\delta\varphi^\ddagger(x)} \frac{\delta Y}{\delta\varphi(x)} \right) d\mu.$$

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  $\{X, Y\} = -\int \left(\frac{\delta X}{\delta\varphi(x)} \frac{\delta Y}{\delta\varphi^{\ddagger}(x)} + (-1)^{|X|} \frac{\delta X}{\delta\varphi^{\ddagger}(x)} \frac{\delta Y}{\delta\varphi(x)} \right) d\mu.$
- The structure $(\Lambda\mathcal{V}, \{.,.\})$ is an example of a Gerstenhaber algebra. To get a BV algebra we need to construct a certain nilpotent differential $\triangle$.

$\star$-product

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$\star$-product

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- Let S_0 be the action of the free (i.e. quadratic) theory. We define the $\star$-product (deformation of the pointwise product):

$$F \star G \doteq m \circ \exp(i\hbar\Gamma_\Delta)(F \otimes G) \ ,$$

where m is the pointwise multiplication and Γ_Δ is the functional differential operator

$$\Gamma_\Delta(F \otimes G) \doteq \frac{1}{2} \int \Delta(x,y) \frac{\delta F}{\delta\varphi(x)} \otimes \frac{\delta G}{\delta\varphi(y)} dx\, dy, \qquad \Delta = \Delta_A - \Delta_R \ ,$$

Δ_R, Δ_A are the retarded and advanced Green functions corresponding to S_0'' seen as a linear operator on $\mathcal{E}$.

Time-ordered product

- The time-ordering operator $\mathcal{T}$ is defined as:

$$\mathcal{T}(F) \doteq e^{i\hbar\Gamma_{\Delta_D}}(F)\,,$$

where $\Gamma_{\Delta_D}F = \int \Delta_D(x,y)\frac{\delta^2 F}{\delta\varphi(x)\delta\varphi(y)}dxdy$ and $\Delta_D = \frac{1}{2}(\Delta_R + \Delta_A)$ is the Dirac propagator.

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- $\mathcal{T}$ allows us to transport the classical structure into the quantum algebra.

Interaction

- For $V \in \mathcal{F}_{\mathrm{reg}}$ the formal S-matrix is defined as the time-ordered exponential:

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- All these structures extend to $\Lambda\mathcal{V}_{\mathrm{reg}}$.

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where $\bigtriangleup$ is explicitely given as:

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- Equivalently: $\hat{s} = \{., S_0 + V\}_{\mathfrak{T}} - i\hbar\bigtriangleup$.

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- To extend $\mathcal{T}^n$ to $\mathcal{T}^n_r$, which is well defined for arbitrary local functionals we use the causal approach of Epstein and Glaser (see the talk of Nicola).
- It was shown in [K. Fredenhagen, KR, 2011] that $\mathcal{T}^n_r$'s define a binary product $\cdot_{\mathcal{T}_r}$ on an appropriate domain.

Renormalized QME and the quantum BV operator

- $\cdot_{\mathcal{T}_r}$ is an associative, commutative product, we can use it in place of $\cdot_{\mathcal{T}}$ and define the renormalized QME and the quantum BV operator as:

$$\{e_{\mathcal{T}_r}^{iV/\hbar}, S_0\} = 0$$

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- These formulas get even simpler if we use the anomalous Master Ward Identity ([Brenecke-Dütsch 08, Hollands 08]):

$$0 = \frac{1}{2}\{V + S_0, V + S_0\}_{\mathcal{T}_r} - \triangle_V ,$$

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where $\triangle_V$ is a local functional depending on the interaction V.

- $\triangle_V^{(1)}$ is well defined on local vector fields in contrast to $\triangle$ and we interpret it as the renormalized BV Laplacian.

Algebraic adiabatic limit

- Assume that we have unitaries $\mathcal{S}(f), f \in \mathcal{D}^n$ with $S(0) = 0$, which generate a *-algebra $\mathfrak{A}$ and satisfy for $f, g, h \in \mathcal{D}$ Bogoliubov's factorization relation

$$\mathcal{S}(f + g + h) = \mathcal{S}(f + g)\mathcal{S}(g)^{-1}\mathcal{S}(g + h)$$

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if the past J_- of $\mathrm{supp}h$ does not intersect $\mathrm{supp}f$.
- We can obtain these as formal S-matrices $\mathcal{S}(f) \doteq \mathcal{S}(V(f))$, discussed previously for a generalized Lagrangian $V(f) = \sum_{j=1}^{n} \int A_j(x) f^j(x) d\mu(x)$, where $f \in \mathcal{D}^n$ and each $A_j(x)$ is a local function on $\mathcal{E}$.

S-matrices

- the map $f \mapsto \mathcal{S}(f)$ induces a large family of objects that satisfy Bogoliubov's factorisation relation, which are labeled by test functions $g \in \mathcal{D}^n$, namely the *relative S-matrices*

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- In the next step we want to remove the restriction to interactions with compact support. Let $G : M \to \mathbb{R}^n$ be smooth and $\mathcal{O}$ be bounded. Set

$$[G]_{\mathcal{O}} = \{g \in \mathcal{D}^n | g \equiv G \text{ on a neighborhood of } J_+(\mathcal{O}) \cap J_-(\mathcal{O})\}\ .$$

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- We consider the $\mathfrak{A}$-valued maps

$$\mathcal{S}_{G,\mathcal{O}}(f) : [G]_{\mathcal{O}} \ni g \mapsto \mathcal{S}_g(f) \in \mathfrak{A} \,.$$

The local algebra $\mathfrak{A}_G(\mathcal{O})$ is defined to be the algebra generated by $S_{G,\mathcal{O}}(f)$, $\mathrm{supp} f \subset \mathcal{O}$.

Relative S-matrices

Crucial results:

- $\mathrm{supp}(g - g') \cap J_-(\mathrm{supp} f) = \varnothing$ implies

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Colloraly (see e.g. [K. Fredenhagen, KR 2015])

The interacting local net $\mathcal{O} \mapsto \mathfrak{A}_G(\mathcal{O})$ is well defined and satisfies the axioms of locality, covariance and the time-slice axiom.

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- Let S_0 be the free generalized Lagrangian and and S_1 the interaction term (including θ). The QME on the level of generalized Lagrangians reads [K. Fredenhagen, K.R. 2011]:

$$e_{\mathcal{T}_r}^{-iS_1/\hbar} \cdot_{\mathcal{T}_r} \left(\{e_{\mathcal{T}_r}^{iS_1/\hbar}, S_0\}_\star\right) \sim 0\,,$$

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- Physical interpretation: invariance ot the S-matrix in the adiabatic limit.

QME in the algebraic adiabatic limit

- The QME can be related to the BRST current J. Classically we have

$$dJ(x) = \sum_{\alpha} \theta_0^{\alpha}(x) \cdot \frac{\delta(S_0 + S_I + \theta)}{\delta\varphi^{\alpha}(x)} ,$$

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- Write $L(f_0,f_1,f_2) = L_0(f_0) + L_I(f_1) + \theta(f_2)$, where $f_1 \equiv 1$ on $\mathrm{supp}\, f_0$ and $f_2 \equiv 1$ on $\mathrm{supp}\, f_1$. Then
$$\{e_{\mathcal{T}_r}^{iV/\hbar}, L_0(f_0)\}_\star = e_{\mathcal{T}_r}^{iV/\hbar} \cdot_{\mathcal{T}_r} dJ(f_2) + \triangle_V\,,$$
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where $V = L_I(f_1) + \theta(f_2)$.
- Result of [Hollands 08]: $\triangle_V$ can be removed, so $R_V(dJ(f_2)) \overset{\text{o.s.}}{=} 0$ (conservation of the current).

Quantum BV operator in the algebraic adiabatic limit

- Assume that $\triangle_V$ can be removed. Define

$$\hat{s}_V(X) \doteq R_V^{-1}\left(\{R_V(X), S_0\} + R_V(dJ(f_2)) \star R_V(X) - R_V(dJ(f_2) \cdot_{\mathcal{T}_\mathrm{r}} X)\right) .$$

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- At this point it is not even clear that $\hat{s}_V$ is nilpotent! However, if we can remove the anomaly $\triangle_V(X)$, then

$$\hat{s}_V(X) = s(X) ,$$

so we reduce the problem to the classical one.

Quantum observables

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- The standard way to proceed is to replace $\hat{s}_V$ with the commutator with the interacting charge Q. It was shown in [KR 13] that

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- However, the opposite implication doesn't work, so the interpretation of this result is unclear.

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- In the presence of gauge theoires one can use the BV formalism combined with the Epstein-Glaser renormalization to construct the theory.
- The gauge invariance of the S-matrix (QME) can be obtained only in the adiabatic limit.
- The standard solution to use the commutator with the conserved charge has a clear mathematical meaning only in the situation where the anomaly $\triangle_V(X)$ can be removed.

Summary

- The algebraic adiabatic limit works fine for theories without gauge symmetries.
- In the presence of gauge theoires one can use the BV formalism combined with the Epstein-Glaser renormalization to construct the theory.
- The gauge invariance of the S-matrix (QME) can be obtained only in the adiabatic limit.
- The standard solution to use the commutator with the conserved charge has a clear mathematical meaning only in the situation where the anomaly $\triangle_V(X)$ can be removed.
- Proposal: take the adiabatic limit seriously and give sense to the renormalized QME in this limit.

Thank you for your attention!