Modular Nuclearity: a generally covariant perspective

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Nuclearity Conditions in QFT

Buchholz and Wichmann (1986) introduced a nuclearity condition in algebraic QFT (Minkowski space): [CMP **106**, 321]

For any $\beta > 0$ and bounded region O, the map

$$\Theta : \mathcal{A}(O) \to \mathcal{H} : A \mapsto e^{-\beta H} A\Omega,$$

is nuclear, where H is the Hamiltonian and Ω the ground state.

This Hamiltonian nuclearity condition

1. encodes good high energy behaviour in bounded spacetime regions, needed for good thermodynamical properties,
2. holds for ground states of the massive free scalar field,
3. was extended to static spacetimes by Verch [LMP **29**, 297 (1993)], and for Dirac fields by D'Antoni, Hollands [CMP **261**, 133 (2006)].

Modular Nuclearity in QFT

The modular nuclearity condition uses a modular operator instead of the Hamiltonian: [Buchholz, D'Antoni, Longo CMP **129**, 115 (1990)]

For any $\alpha \in (0, \frac{1}{2})$ and bounded regions $\tilde{O} \Subset O$, the map

$$\Xi : \mathcal{A}(\tilde{O}) \to \mathcal{H} : A \mapsto \Delta^{\alpha} A\Omega$$

is nuclear, where Δ is a modular operator of Ω for $\mathcal{A}(O)$.

The physical interpretation of Δ is unclear, but modular nuclearity

1. is equivalent to Hamiltonian nuclearity, in the case of ground states, because the spectra of Δ and H are related,
2. has a nice physical consequence: the split property,
3. was used in constructive 2D integrable models to find local observables [Buchholz and Lechner, AHP **5**, 1065 (2004)],
4. can be generalised to other states and curved spacetimes.

Why use Modular Operators?

To avoid some of the following problems:

1. Hamiltonians only exist in special spacetimes and for special states, and they contain global information,
2. locally smeared energy densities are a good alternative, but they are difficult to handle, even for quasi-free states of a free scalar field (not of second quantised form, possibly not essentially self-adjoint),
3. the Hadamard condition is local, generally covariant and encodes finite energy density for free fields, but is hard to generalise to an axiomatic setting Verch [CMP **205**, 337(1999)].

Questions about Modular Nuclearity

What we would like to know about modular nuclearity:

1. How far can we generalise the statement of the property?
2. How does it behave in the light of locality and general covariance?
3. What does it mean physically?
4. For a free scalar field, how many states satisfy it?
5. Does it reduce to the Hadamard condition for free fields?

Modular Operators

To any state $\omega : \mathcal{A} \to \mathbb{C}$ one can associate:

- a GNS-representation $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$,
- an orthogonal projection $Q_\omega \in \pi_\omega(\mathcal{A})''$ onto the subspace

$$\mathcal{H}'_\omega := \overline{\pi_\omega(\mathcal{A})'\Omega_\omega},$$

- a Tomita operator $S_\omega := J_\omega \Delta_\omega^{\frac{1}{2}}$ (typically unbounded) with

$$S_\omega Q_\omega A \Omega_\omega := Q_\omega A^* \Omega_\omega, \qquad A \in \pi_\omega(\mathcal{A})''.$$

The modular operator Δ_ω is extended by 0 on $(\mathcal{H}'_\omega)^\perp$.

Examples:

1. When ω is pure, Q_ω projects onto the span of Ω_ω, $\Delta_\omega = Q_\omega$.
2. When Ω_ω is cyclic and separating, $Q_\omega = 1$. (Reeh-Schlieder)

The l^p Property

Instead of a nuclearity condition, we impose (stronger) l^p conditions:

For any map $\Xi : B_1 \to B_2$ between Banach spaces, $n \in \mathbb{N}$,

$$\alpha_n(\Xi) := \inf_{\Xi_n \text{ of rank } \leq n} \|\Xi - \Xi_n\|$$

is the n^{th} approximation number. Ξ is called an l^p-map for $p > 0$ iff the p-quasi-norm is finite:

$$\|\Xi\|_p := \left(\sum_{n=1}^{\infty} \alpha_n(\Xi)^p \right)^{\frac{1}{p}} < \infty.$$

1. For Hilbert spaces B_i, $\alpha_n(\Xi)$ are the decreasing eigenvalues of $|\Xi|$,
2. l^p-maps are compact and they build a linear space,
3. the smaller p, the stronger the requirement.

Estimates for $\|.\|_p$

The p-quasi-norms satisfy

$$
\begin{aligned}
\|\Xi + \Theta\|_p &\leq \max\{2, 2^{\frac{2}{p}-1}\}(\|\Xi\|_p + \|\Theta\|_p) \\
\|\Xi \cdot \Theta\|_p &\leq \|\Xi\| \cdot \|\Theta\|_p \\
\|\Xi \cdot \Theta\|_r &\leq \|\Xi\|_p \cdot \|\Theta\|_q, \qquad r^{-1} = p^{-1} + q^{-1}.
\end{aligned}
$$

To obtain new l^p maps from old ones we will often use

Lemma

If $\Xi_2 : B_1 \to B_2$ *and* $\Xi_3 : B_1 \to B_3$ *are linear maps such that*

$$\|\Xi_3(b)\| \leq \|\Xi_2(b)\|, \qquad b \in B_1,$$

then $\|\Xi_3\|_p \leq \|\Xi_2\|_p$ *for all* $p > 0$.

Definition of Modular Nuclearity

A generally covariant QFT (GCQFT) is a functor: $M \to \mathcal{A}(M)$.

A state ω on $\mathcal{A}(M)$ satisfies the modular nuclearity condition iff for all causal embeddings $\tilde{O} \Subset O \Subset M$ with bounded range and all $\alpha \in (0, \frac{1}{2})$, the following maps are l^p for all $p > 0$:

$$\Xi_\alpha : \mathcal{A}(\tilde{O}) \to \mathcal{H}_{\omega|_O} : A \mapsto \Delta^\alpha_{\omega|_O} \pi_{\omega|_O}(A) \Omega_{\omega|_O},$$

where $\omega|_O$ is the restriction of ω to $\mathcal{A}(O)$.

Our definition:

- differs slightly from previous ones,
- makes sense for all states in C^*-algebraic GCQFT's,
- is preserved under pull-backs,
- may be supplemented by estimates on $\|\Xi_\alpha\|_p$.

A Useful Estimate

Shrinking $\tilde{O}$ cannot increase $\|\Xi_\alpha\|_p$. The same holds for enlarging O:

Lemma

Let $\mathcal{B} \subset \mathcal{A}$ be an inclusion of C^*-algebras and ω be a state on $\mathcal{A}$. Then

$$\|\Delta_\omega^\alpha \pi_\omega(b)\Omega_\omega\| \leq \|\Delta_{\omega|_\mathcal{B}}^\alpha \pi_{\omega|_\mathcal{B}}(b)\Omega_{\omega|_\mathcal{B}}\|, \qquad b \in \mathcal{B}.$$

Main ingrediens of the proof:

- The GNS-representation of $\omega|_\mathcal{B}$ is contained in that of ω.
- We have $\Delta_\omega \leq \Delta_{\omega|_\mathcal{B}}$ on the form domain of $\Delta_{\omega|_\mathcal{B}}$, and
- $x \mapsto x^\beta$ is operator monotone on $x \geq 0$ for $\beta \in [0,1]$, by Löwner's Theorem.
- Tricky part: $\mathcal{H}_{\omega|_\mathcal{B}} \subsetneq \mathcal{H}_\omega$ in general, so $\Delta_{\omega|_\mathcal{B}}$ and Δ_ω act on different spaces. Extending $\Delta_{\omega|_\mathcal{B}}$ by 0 may not preserve the estimate.

Convex Combinations

Let ω_1, ω_2 be two states on $\mathcal{A}$ and $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$.

Proposition

For $\omega := \lambda_1\omega_1 + \lambda_2\omega_2$ *and* $\alpha \in (0, \frac{1}{2})$*:*

$$\|\Delta_\omega^\alpha \pi_\omega(a)\Omega_\omega\| \leq \sum_{i=1}^{2} \sqrt{\lambda_i}\|\Delta_{\omega_i}^\alpha \pi_{\omega_i}(a)\Omega_{\omega_i}\|.$$

Proof:
We may identify $\pi_\omega(a) := \pi_{\omega_1}(a) \oplus \pi_{\omega_2}(a)$ in $\mathcal{H} := \mathcal{H}_{\omega_1} \oplus \mathcal{H}_{\omega_2}$, with $\Omega_\omega := \sqrt{\lambda_1}\Omega_{\omega_1} \oplus \sqrt{\lambda_2}\Omega_{\omega_2}$. We then have $\pi_\omega(\mathcal{A}) \subset \mathcal{R}$ for the algebra

$$\mathcal{R} := \pi_{\omega_1}(\mathcal{A})'' \oplus \pi_{\omega_2}(\mathcal{A})''.$$

Note that $(\mathcal{R}, \Omega)$ has $\Delta = \Delta_{\omega_1} \oplus \Delta_{\omega_2}$ and apply the previous lemma.

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$$\|\Delta_\omega^\alpha \pi_\omega(a)\Omega_\omega\| \leq \sum_{i=1}^{2} \sqrt{\lambda_i} \|\Delta_{\omega_i}^\alpha \pi_{\omega_i}(a)\Omega_{\omega_i}\|.$$

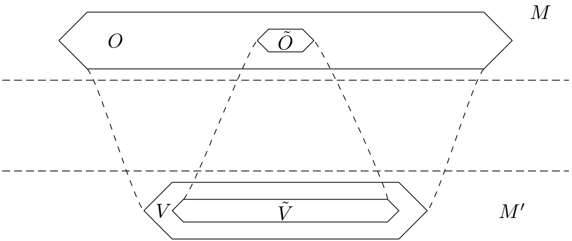
Corollary

Modular nuclearity is preserved under convex combinations, with

$$\|\Xi_{\alpha;\omega}\|_p \leq \max\{2, 2^{\frac{2}{p}-1}\} \sum_{i=1}^{2} \sqrt{\lambda_i} \|\Xi_{\alpha;\omega_i}\|_p.$$

Spacetime Deformation

Using the time-slice axiom we can perform spacetime deformations:



Given $\tilde{O} \Subset O \Subset M$ we can deform M in the past to some spacetime M' and find $\tilde{V} \Subset V \Subset M'$ as shown.

The causal propagation then implies

Theorem

Any state ω on $\mathcal{A}(M')$ with modular nuclearity for $\tilde{V} \Subset V \Subset M'$ propagates to a state on $\mathcal{A}(M)$ with modular nuclearity for $\tilde{O} \Subset O \Subset M'$.

Second Quantisation

Consider a massive free scalar field with Weyl algebra $\mathcal{W}(M)$.

- For a quasi-free state ω on $\mathcal{W}(O)$, $O \Subset M$, the operators

$$S_\omega = \Gamma(s_\omega), \qquad J_\omega = \Gamma(j_\omega), \qquad \Delta_\omega = \Gamma(\delta_\omega)$$

  are second quantised, with $s_\omega = j_\omega \delta_\omega$.
- In the one-particle Hilbert space $\mathcal{H}_{1,O}$ we have

$$s_\omega \phi(f)\Omega_\omega = q_\omega \phi(\bar{f})\Omega_\omega$$

  for all $f \in C_0^\infty(O)$ and a projection q_ω onto a subspace $\mathcal{H}'_{1,O}$.
- For any $\tilde{O} \Subset O$ we let $\tilde{E}$ denote the real orthogonal projection onto

$$\tilde{H} := q_\omega C_0^\infty(\tilde{O})\Omega_\omega.$$

Second Quantisation – II

δ_ω is typically unbounded, but $s_\omega \tilde{E} = \tilde{E}$, so

$$\|\delta_\omega^{\frac{1}{4}} \tilde{E}\|^2 = \|\tilde{E}\delta_\omega^{\frac{1}{2}} \tilde{E}\| = \|\tilde{E} j_\omega \tilde{E}\| \leq 1.$$

We also have $\|\delta_\omega^\alpha \tilde{E}\| \leq 1$ for $\alpha \in [0, \frac{1}{2}]$. Moreover:

Theorem

For all $\alpha \in (0, \frac{1}{2})$ the following two statements are equivalent:

1. *For all $p > 0$ the following map is l^p:*

$$\Xi_\alpha : \mathcal{W}(\tilde{O}) \to \mathcal{H}_{\omega|_O} : A \mapsto \Delta_{\omega|_O}^\alpha \pi_{\omega|_O}(A)\Omega_{\omega|_O}.$$

2. *For all $p > 0$ the real linear operator $\delta_\omega^\alpha \tilde{E}$ is l^p on $\mathcal{H}_{1,O}$.*

This reduces the problem to one-particle modular nuclearity.

Relation to the Symplectic Form

- Using ω_{2+} we define the Hilbert space

$$\mathcal{K}_O = \overline{C_0^\infty(O)/\{\omega_{2+}(\bar{f}, f) = 0\}}.$$

  For $O \Subset M'$, P_O projects orthogonally onto $\mathcal{K}_O \subset \mathcal{K}_M$.
- The symplectic form $\sigma = -2i\omega_{2-}$ is expressed by an operator

$$\Sigma_O : \mathcal{K}_O \to \mathcal{K}_O, \qquad \langle f, \Sigma f' \rangle = \frac{i}{2}\sigma(\bar{f}, f').$$

  $\Sigma_O = P_O \Sigma_M P_O$ is bounded.

Proposition

One-particle modular nuclearity is equivalent to

$$(P_O - \Sigma_O^2)^\beta P_{\tilde{O}} \ : \ \mathcal{K} \to \mathcal{K}$$

being l^p iff $\beta, p > 0$, for all bounded diamond regions $\tilde{O} \Subset O \Subset M$.

Modular Nuclearity for Ground States

In ground states, $\Sigma_M = \mathrm{sign}(H)$ with Hamiltonian H related to $-\Delta + m^2$ on a Cauchy surface $\mathcal{C}$. Note:

$$1 - \Sigma_M^2 = 0, \text{ but } P_O - \Sigma_O^2 \neq 0 \text{ in general.}$$

To establish one-particle nuclearity we proved:

Theorem

For any complete Riemannian manifold $\mathcal{C}$, $m > 0$, $a, b, c \in \mathbb{R}$ and $\tilde{\chi}, \chi \in C_0^\infty(\Sigma)$ with $\chi \equiv 1$ on a nbhd. of $\mathrm{supp}(\tilde{\chi})$,

$$(-\Delta + m^2)^a (1 - \chi)(-\Delta + m^2)^b \tilde{\chi}\, (-\Delta + m^2)^c.$$

is l^p on $L^2(\mathcal{C})$ for all $p > 0$.

The proof uses finite propagation speeds. [Cheeger, Gromov, Taylor J.Diff.Geom. **17**, 15 (1982)]

Quasi-free Hadamard States

For general quasi-free Hadamard states:

- In static spacetimes, we improved the analysis of their local quasi-equivalence. [Verch, CMP **160**, 507 (1994)]
  The Hadamard property implies modular nuclearity (but not vice-versa!)
- In general globally hyperbolic space times, we used spacetime deformation to reduce to the static case.

Our main result:

Theorem

A quasi-free Hadamard state of a free scalar quantum field in any globally hyperbolic space-time (with any mass, scalar curvature coupling or external potential energy) satisfies modular nuclearity.

Conclusions and Outlook

Modular nuclearity

- is well-defined and well-behaved for generally covariant QFT,
- for quasi-free states is equivalent to one-particle modular nuclearity, related to the symplectic form, and
- strictly weaker than the Hadamard condition for free scalar fields.

Future directions:

- When is modular nuclearity equivalent to Hamiltonian nuclearity?
- Can we obtain good estimates for $\|\Xi_\alpha\|_p$?
- Apply these estimates to estimate entanglement entropy.
- Does modular nuclearity help to construct non-trivial GCQFT's?