# On the improper ground state of Nelson's Massless Model

Vincent Beaud

Institute for Theoretical Physics
ETH Zürich

October 17, 2015

*Work in progress with G.M. Graf*

"QFT: Infrared problems and constructive aspects", TU Munich

# IR catastrophe illustration: van Hove model

van Hove Hamiltonians (van Hove '52, Dereziński '03) are a class of quadratic Hamiltonians on $\mathcal{F}$ with linear perturbation in $a$ and $a^*$:

$$\begin{aligned} H_{\mathrm{vh}} &= \mathrm{d}\Gamma(\omega) + a^*(\omega f) + a(\omega f) \\ &= \int_{\mathbb{R}^3} \mathrm{d}k\, \omega(k)\big(a^*(k) + \overline{f(k)}\big)\big(a(k) + f(k)\big) - \|\omega^{1/2} f\|_2^2 \,. \end{aligned}$$

Assumptions:

- $\omega(k) = |k|$ (massless field),
- $\omega^\alpha f \in L^2(\mathbb{R}^3, \mathrm{d}k)$ for $\alpha = \frac{1}{2}, 1$.

If $f \in L^2(\mathbb{R}^3, \mathrm{d}k)$, apply the Weyl transformation

$$W(f) H_{\mathrm{vh}} W(f)^* = H_{\mathrm{B}} - \|\omega^{1/2} f\|_2^2 \,.$$

$H_{\mathrm{vh}}$ has ground state $W(f)^*\Omega$ (coherent state).
Mean number of photons in ground state is $\|f\|_2^2$ (IR catastrophe).

# Nelson's massless model (N '64, Fröhlich '73, Pizzo '03)

One spinless non-relativistic particle (electron) coupled to a massless bosonic radiation field (photons): $\mathcal{H} = L^2(\mathbb{R}^3, \mathrm{d}x) \otimes \mathcal{F}$. Hamiltonian:

$$H = \frac{p^2}{2} \otimes 1 + 1 \otimes H_{\mathrm{B}} + g \int_{\mathbb{R}^3} \frac{\mathrm{d}k}{\sqrt{2|k|}} \big( e^{-ik\cdot x} \otimes a^*(k) + e^{ik\cdot x} \otimes a(k) \big) \kappa(k)$$

with $p = -i\nabla_x$, $H_{\mathrm{B}} = \mathrm{d}\Gamma(\omega)$, $\omega(k) = |k|$ and $\kappa$ an UV cutoff (form factor of the electron).

- Interaction is infinitesimal wrt free Hamiltonian $H_0$
  $\Rightarrow H$ is self-adjoint on $D(H_0)$ and bounded below.
- $H$ is translation invariant: $[H, P] = 0$ for $P = p \otimes 1 + 1 \otimes P_{\mathrm{B}}$.
- Fibration of the Hilbert space: $\exists$ a unitary $I$ such that

$$\mathcal{H} = I^* \Big( \int_{\oplus} \mathrm{d}\xi \, \mathcal{H}(\xi) \Big) I \,, \qquad \mathcal{H}(\xi) \cong \mathcal{F} \,.$$

Idea: $L^2\big(\mathbb{R}^3_x, \mathcal{F}\big) \rightsquigarrow L^2\big(\mathbb{R}^3_\xi, \mathcal{F}\big)$.

# Nelson's massless model: Fibre decomposition

Hamiltonian on $\mathcal{H}(\xi)$:

$$H(\xi) = \frac{1}{2}(\xi - P_{\mathrm{B}})^2 + H_{\mathrm{B}} + g \int_{\mathbb{R}^3} \frac{\mathrm{d}k}{\sqrt{2|k|}} \big(a^*(k) + a(k)\big)\kappa(k)$$

$$= \frac{1}{2}(\xi - P_{\mathrm{B}})^2 + H_{\mathrm{B}} + a^*(\omega g) + a(\omega g)\,,$$

with $P_{\mathrm{B}} = \int \mathrm{d}k\, k\, a^*(k)a(k)$, $\omega(k) = |k|$ and $g(k) = g\dfrac{\kappa(k)}{\sqrt{2|k|^3}}$.

Assumptions and properties:

- Restrict to $|\xi| < 1 - 2a$ for (small) $a > 0$ (non-relativistic electron).
- $\omega^\alpha g \in L^2(\mathbb{R}^3, \mathrm{d}k)$ for $\alpha = \frac{1}{2}, 1$, but $g \notin L^2(\mathbb{R}^3, \mathrm{d}k)$.
- For $\xi \neq 0$, $H(\xi)$ has purely ac spectrum ($\Rightarrow$ no ground state) (Fröhlich '73, Hasler-Herbst '08)

# Nelson's massless model: improper ground state

Hamiltonian on $\mathcal{H}(\xi)$:

$$H(\xi) = \frac{1}{2}(\xi - P_{\mathrm{B}})^2 + H_{\mathrm{B}} + a^*(\omega g) + a(\omega g)\,,$$

with $\omega(k) = |k|$ and $g(k) = g\dfrac{\kappa(k)}{\sqrt{2|k|^3}}$.

- Assume anyway: $\exists$ ground state $\Phi(\xi)$: $H(\xi)\Phi(\xi) = E(\xi)\Phi(\xi)$ and it is coherent for $|k| \to 0$, i.e. $a(k)\Phi(\xi) \sim -f(k)\Phi(\xi)$, then

$$f(k) \underset{|k|\to 0}{\approx} g\frac{\kappa(k)}{\sqrt{2|k|}\big(|k| - k\cdot v\big)}\,, \qquad v = \nabla E(\xi)\,.$$

- For $|k| \to 0$, ground state has formal expression $\Phi(\xi) = W(f)^*\Omega$.
- $f \notin L^2(\mathbb{R}^3, \mathrm{d}k) \Rightarrow \Phi(\xi)$ is not normalizable (improper).

Goal: $H(\xi)$ has ground state after (formal) Weyl transformation $W(f)$. (Fröhlich '73, Pizzo '03)

# Existence of ground state: Step 1

Set IR cutoffs $\sigma > 0$ on interaction and $\sigma \geqslant \tau \geqslant 0$ on Fock space:

$$H_\tau^\sigma(\xi) = \frac{1}{2}(\xi - P_{\mathrm{B},\tau})^2 + H_{\mathrm{B},\tau} + a^*(\omega g_\sigma) + a(\omega g_\sigma)\,,$$

and $H_\sigma(\xi) \equiv H_\sigma^\sigma(\xi)$.

Let $\frac{1}{2} < q < 1$ be arbitrary, but fixed along with (small) $a > 0$.

**Theorem**

*For sufficiently small $g$ (depending on $a$ and $q$, but not on $\sigma$), $H_\sigma(\xi)$ has a non-degenerate ground state $\big(E_\sigma(\xi), \mathrm{P}_\sigma(\xi)\big)$ with gap*

$$\Delta_\sigma(\xi) \geqslant qa\sigma\,,$$

*for all $\sigma > 0$ and $|\xi| \leqslant 1 - 2a$. $(E_\sigma(\xi), \mathrm{P}_\sigma(\xi))$ are real-analytic in such $\xi$.*

The proof is inductive in $\sigma = \sigma^*/2^n$ for $\sigma^*$ large enough and $n \in \mathbb{N}_0$.

# Existence of ground state: Step 1

Base case: $g_\sigma \equiv 0$ for large $\sigma > 0 \quad \Rightarrow \quad$ ground state is vacuum $\Omega_\sigma$.

Induction step: $H_\sigma(\xi)$ has GS $\Psi_\sigma(\xi)$ with energy $E_\sigma(\xi)$ and gap $\geqslant qa\sigma$.
$\sigma \rightsquigarrow \sigma/2 = \tau$.

- Lower cutoff of Fock space: $H_\sigma(\xi) \equiv H_\sigma^\sigma(\xi) \rightsquigarrow H_\tau^\sigma(\xi)$.
  GS $\Omega_{\tau,\sigma} \otimes \Psi_\sigma(\xi)$ with energy $E_\sigma(\xi)$ and gap $\geqslant a\tau$.

  Only need: $$E_\sigma(\xi - p) + h - E_\sigma(\xi) \geqslant \frac{3}{2} ah\,, \quad (|p| \leqslant h).$$

- Lower cutoff of interaction: $H_\tau^\sigma(\xi) \rightsquigarrow H_\tau(\xi)$.
  $E_\sigma(\xi) + qa\tau$ is in a gap for $H_\tau^\sigma(\xi)$ and $E_\tau(\xi) \leqslant E_\sigma(\xi)$.

  Show that $E_\sigma(\xi) + qa\tau$ is still in a gap for $H_\tau(\xi)$.

  $$A + B - z = (A - z)^{1/2}\big(1 + (A - z)^{-1/2} B (A - z)^{-1/2}\big)(A - z)^{1/2}$$

  is still invertible if $\|(A - z)^{-1/2} B (A - z)^{-1/2}\| < 1$.

# Existence of ground state: Step 2

Define the transformed Hamiltonian

$$H_\sigma^{\mathrm{W}}(\xi, v) := W(g_{\sigma,v}) H_\sigma(\xi) W(g_{\sigma,v})^*,$$

with

$$g_{\sigma,v}(k) = g \frac{\kappa_\sigma(k)}{\sqrt{2|k|}\big(|k| - k\cdot v\big)} \in L^2(\mathbb{R}^3, \mathrm{d}k).$$

- For $v = \nabla E_\sigma(\xi)$ and $\sigma = 0$ we have $g_{\sigma,v} \equiv f$.
- $H_\sigma^{\mathrm{W}}(\xi, v)$ has ground state energy $E_\sigma(\xi)$ with gap $\geqslant qa\sigma$ and associated projection $\mathrm{P}_\sigma^{\mathrm{W}}(\xi, v)$.

**Theorem**

*Let $\sigma(n) = \sigma/2^n$. For sufficiently small $g$, the limit*

$$\mathrm{P}^{\mathrm{W}}(\xi) := \lim_{n\to\infty} \mathrm{P}_{\sigma(n)}^{\mathrm{W}}\big(\xi, \nabla E(\xi)\big)$$

*exists for all $|\xi| \leqslant 1 - 2a$ and is independent of $\sigma$. It satisfies $\big(H^{\mathrm{W}}(\xi) - E(\xi)\big)\mathrm{P}^{\mathrm{W}}(\xi) = 0$, where $H^{\mathrm{W}} \equiv H_{\sigma=0}^{\mathrm{W}}$ and $E \equiv E_{\sigma=0}$.*

# Step 2: Estimates on $\mathrm{P}^{\mathrm{W}}_\sigma(\xi, v)$ and $\nabla E_\sigma(\xi)$

The proof relies on the following two estimates for any $\sigma > 0$

**Theorem**

*Let $\tau = \sigma/2$. There exists $C > 0$ (independent of $\sigma$ and $g$) such that for small enough $g$ and under suitable assumptions*

(i) $\|\mathrm{P}^{\mathrm{W}}_\sigma(\xi, v) - \mathrm{P}^{\mathrm{W}}_\tau(\xi, v)\| \leqslant C g \sigma^{(1-\alpha)/2}$,

(ii) $|\nabla E_\sigma(\xi) - \nabla E_\tau(\xi)| \leqslant C g \sigma^{(1-\alpha)/2}$,

*for all $|\xi| \leqslant 1 - 2a$ and some $0 < \alpha < 1$.*

If (i) holds for all $\sigma > 0$ and $v = \nabla E(\xi)$, then convergence follows.

What are the "suitable assumptions"? The first is

$$v \in \mathcal{I}_\sigma \quad \Leftrightarrow \quad |v - \nabla E_\sigma(\xi)| \leqslant c\sigma^{(1-\alpha)/2}$$

for some $0 < \alpha < 1$ and (small) $c > 0$.

Hence, $\nabla E(\xi) \in \mathcal{I}_\sigma$ for all $\sigma > 0$ and small enough $g$.

# Step 2: Motivation of the requirements

The second "suitable assumption" is an estimate of the form

$$\|\big(H_\sigma(\xi) - E_\sigma(\xi)\big)^{-1/2}\big(\xi - P_{\mathrm{B},\sigma} - \nabla E_\sigma(\xi)\big)\Psi_\sigma(\xi)\|_2 \leqslant \sqrt{2}\sigma^{-\alpha/2}\,.$$

Motivation: set $\mathrm{P}_1 \equiv \mathrm{P}^{\mathrm{W}}_\sigma(\xi, v)\,, \mathrm{P}_2 \equiv \mathrm{P}^{\mathrm{W}}_\tau(\xi, v)$ and estimate $\|\mathrm{P}_1 - \mathrm{P}_2\|$.

- For any orthogonal projections $\mathcal{P}_i = \psi_i(\psi_i, \cdot)$ $(i = 1, 2)$

$$\|\mathcal{P}_1 - \mathcal{P}_2\|^2 \leqslant 2\,\mathrm{tr}\big(\mathcal{P}_1(\mathcal{P}_1 - \mathcal{P}_2)^2\mathcal{P}_1\big) = 2\|\big(\mathcal{P}_1 - \mathcal{P}_2\big)\psi_1\|_2^2\,.$$

- $H_i$ have gaps $O(\sigma)$ and $|E_1 - E_2| = O(g^2\sigma)$
  $\Rightarrow \exists\gamma \subset \mathbb{C} \setminus \cup\{\mathrm{spec}\, H_i\}$ encircling only $E_\sigma(\xi)$ and $E_\tau(\xi)$.

$$\mathrm{P}_i = -\frac{1}{2\pi i}\int_\gamma \mathrm{d}z\,\big(H_i - z\big)^{-1}\,,$$

$$\big(\mathrm{P}_1 - \mathrm{P}_2\big)\Psi_1 = \frac{1}{2\pi i}\int_\gamma \mathrm{d}z\,\big(H_2 - z\big)^{-1}\big(H_1 - H_2\big)\Psi_1\frac{\mathrm{d}z}{E_1 - z}\,.$$

# Step 2: Motivation of the requirements

- $\|\mathrm{P}_1 - \mathrm{P}_2\| \leqslant C \sup_{z\in\gamma} \|\big(H_2 - z\big)^{-1}\big(H_1 - H_2\big)\Psi_1\|_2$
- The two most serious terms in $H_1 - H_2$ have the form

$$\big(f_0 + a^*(f)\big)\cdot\big(\xi - W(g_{\sigma,v})P_{\mathrm{B},\tau}W(g_{\sigma,v})^* - v\big)$$

  with $|f_0| = O(g^2\sigma)$ and $\|f\|_2^2 = O(g^2\sigma^2)$.
- For the first, we have the bound $C'g^2$ times

$$\sigma\|\big(\widetilde{H}_2 - z\big)^{-1}\big(\xi - P_{\mathrm{B},\sigma} - v\big)\Psi_\sigma(\xi)\|_2$$
$$\leqslant |v - \nabla E_\sigma(\xi)| + \sigma^{1/2}\|\big(\widetilde{H}_2 - z\big)^{-1/2}\big(\xi - P_{\mathrm{B},\sigma} - \nabla E_\sigma(\xi)\big)\Psi_\sigma(\xi)\|_2\,,$$

- Hence the requirements $|v - \nabla E_\sigma(\xi)| \leqslant c\sigma^{(1-\alpha)/2}$ and

$$\|\big(H_\sigma(\xi) - E_\sigma(\xi)\big)^{-1/2}\big(\xi - P_{\mathrm{B},\sigma} - \nabla E_\sigma(\xi)\big)\Psi_\sigma(\xi)\|_2 \leqslant \sqrt{2}\sigma^{-\alpha/2}\,.$$

  Taking the square and setting $\Pi = \xi - P_{\mathrm{B},\sigma} - \nabla E_\sigma(\xi)$, we get

$$H_\sigma(\xi) - \frac{1}{2}\sigma^\alpha \Pi \mathrm{P}_\sigma(\xi)\Pi \geqslant E_\sigma(\xi)\,.$$

# Step 2: Auxiliary Hamiltonians

Need to introduce auxiliary Hamiltonians

$$h_\sigma(\xi, v) := H_\sigma(\xi) - \frac{1}{2}\sigma^\alpha \big(\xi - P_{\mathrm{B},\sigma} - v\big)\mathrm{P}_\sigma(\xi)\big(\xi - P_{\mathrm{B},\sigma} - v\big)\,.$$

**Theorem**

*There exist $c > 0$ such that, for all sufficiently small $g$, $h_\sigma(\xi, v)$ has a non-degenerate ground state $\big(E_\sigma(\xi, v), \mathrm{P}_\sigma(\xi, v)\big)$ with gap*

$$\Delta_\sigma(\xi, v) \geqslant qa\sigma$$

*for all $\sigma > 0$, $|\xi| \leqslant 1 - 2a$ and $|v - \nabla E_\sigma(\xi)| \leqslant c\sigma^{(1-\alpha)/2}$. For $v = \nabla E_\sigma(\xi)$, the ground state is that of $H_\sigma(\xi)$.*

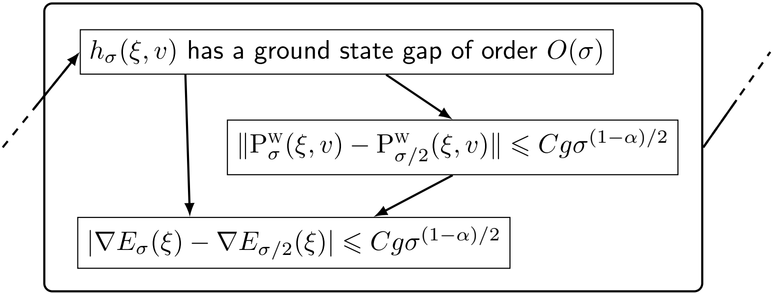
The proof is again by induction in $\sigma = \sigma^*/2^n$ for large $\sigma^*$ and $n \in \mathbb{N}_0$.

# Step 2: The induction step scheme

Base case: $g_\sigma \equiv 0$ for large $\sigma > 0 \quad \Rightarrow \quad$ ground state is vacuum $\Omega_\sigma$.

Induction step: $\sigma \rightsquigarrow \sigma/2$. The estimates on $\mathrm{P}^{\mathrm{W}}_\sigma(\xi, v)$ and $\nabla E_\sigma(\xi)$ are proven inductively according to the following diagram

Induction step $\sigma \to \sigma/2$



# Discussion

The existence of a ground state for $H^{\mathrm{W}}_\sigma(\xi, \nabla E_\sigma(\xi))$ in the limit $\sigma \to 0$ has already been proven by

- J. Fröhlich in '73 (functionals on some algebra of operators),
- A. Pizzo in '03 (ground eigenvector).

Both used IR cutoffs $\sigma$ and the existence of isolated eigenvalue for $\sigma > 0$.

The present proof uses ground state projections. Differences:

- Improved estimate: $E_\sigma(\xi - p) + h - E_\sigma(\xi) \geqslant \frac{3}{2} a h \,, \quad (|p| \leqslant h)$.
- Working with projections avoids the possible trouble $\Psi_\sigma \to 0$.
- Estimates proven for neighbourhoods $|v - \nabla E_\sigma(\xi)| = O(\sigma^{(1-\alpha)/2})$.
- No upper bound on the UV cutoff.
- $|\xi| \leqslant 1 - 2a$ with $a$ arbitrarily small for small enough $g$.

Hope: analyticity of $\mathrm{P}^{\mathrm{W}}(\xi)$ in some neighbourhood of $0$?

# Summary

- Translation invariant Nelson's massless model.
  Total momentum fibration: $H$ on $L^2(\mathbb{R}^3) \otimes \mathcal{F} \rightsquigarrow H(\xi)$ on $\mathcal{F}$.
- $H(\xi)$ has pure ac spectrum $\Rightarrow$ no ground state.
- IR regular $H_\sigma(\xi)$ has ground state for $\sigma > 0$.
- Weyl-transformed $H^{\mathrm{W}}_\sigma(\xi, v)$ has ground state for $\sigma \geqslant 0$ if $v = \lim_{\sigma \to 0} \nabla E_\sigma(\xi)$.
- Proof essentially relies on two estimates

  (i) $\|\mathrm{P}^{\mathrm{W}}_\sigma(\xi, v) - \mathrm{P}^{\mathrm{W}}_\tau(\xi, v)\| \leqslant C g \sigma^{(1-\alpha)/2}$,

  (ii) $|\nabla E_\sigma(\xi) - \nabla E_\tau(\xi)| \leqslant C g \sigma^{(1-\alpha)/2}$,

  and that

  $$h_\sigma(\xi, v) = H_\sigma(\xi) - \frac{1}{2}\sigma^\alpha(\xi - P_{\mathrm{B},\sigma} - v)\mathrm{P}_\sigma(\xi)(\xi - P_{\mathrm{B},\sigma} - v)$$

  has the same ground state and gap as $H_\sigma(\xi)$ for $v = \nabla E_\sigma(\xi)$.

Thank you for your attention!