The universal C*-algebra of the electromagnetic field

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Motivation

Critique of AQFT:

The connection between the key concepts of AQFT and LQFT, i.e. observables and quantum fields, is not at all clear.
M. Kuhlmann [2000]

In AQFT the algebra of observables contains the physical content. This has not been extended to include local gauge theory.
E. MacKinnon [2005]

To be lured away from the Standard Model by AQFT is sheer madness.
D. Wallace [2010]

To some questions (e.g., is there a photon field that interacts with an electron field?), LQFT provides the obvious right answer and AQFT cannot.
D.J. Baker [2015]

AQFT has given us a frame and a language, not a theory.
R. Haag [1996]

Is AQFT capable of describing concrete physical systems?

Example: electromagnetic field in $d = 4$. It may interact with ...

- nothing (asymptotic radiation)
- classical currents (external electromagnetic forces)
- quantum currents (atoms, ions)
- electrons, protons (elementary particles)
- quarks (constituents) ...

LQFT approach: write down in each instance a classical Lagrangian describing the interaction and "quantize" it

AQFT approach: construct a universal algebra of the electromagnetic field and "represent" it (as the case may be)

Message of talk: There is such a universal C*-algebra!

Electromagnetic field

Notation:

- $\mathcal{D}_r(\mathbb{R}^4)$ space of real, tensor-valued test functions with compact support, rank $r = 0, \ldots, 4$, totally anti-symmetric (forms)
- $d : \mathcal{D}_r(\mathbb{R}^4) \to \mathcal{D}_{r+1}(\mathbb{R}^4)$ exterior derivative (generalized curl)
- $\delta : \mathcal{D}_r(\mathbb{R}^4) \to \mathcal{D}_{r-1}(\mathbb{R}^4)$ co-derivativ (generalized divergence)

Distinctive properties of the electromagnetic field F:

- $F : \mathcal{D}_2(\mathbb{R}^4) \to \mathfrak{P}$ (linear map to generators of some *-algebra $\mathfrak{P}$)
- $F(\delta h) = 0, \; h \in \mathcal{D}_3(\mathbb{R}^4)$ (homogeneous Maxwell equation)
- $j(g) \doteq F(dg), \; g \in \mathcal{D}_1(\mathbb{R}^4)$ (inhomogeneous Maxwell equation)
- $F(f) \mapsto F(f_P), \; P \in \mathcal{P}_+^\uparrow$ (covariance)
- $[F(f_1), F(f_2)] = 0$ if $\operatorname{supp} f_1 \perp \operatorname{supp} f_2$ (locality)

Convenient to proceed from F to "intrinsic vector potential" A:

$$A(\delta f) \doteq F(f)\,, \quad f \in \mathcal{D}_2(\mathbb{R}^4)$$

Note: $\delta(\delta f) = 0$, $f \in \mathcal{D}_2(\mathbb{R}^4)$, *i.e.* $\delta f \in \mathcal{D}_1(\mathbb{R}^4)$ is "co-closed".

Question: Are all co-closed elements of $\mathcal{D}_1(\mathbb{R}^4)$ of this form?

Local Poincaré Lemma

Let $g \in \mathcal{D}_1(\mathbb{R}^4)$ be co-closed, $\delta g = 0$, and $\operatorname{supp} g \subset \mathcal{O}$ (double cone). There is $f \in \mathcal{D}_2(\mathbb{R}^4)$ with $\operatorname{supp} f \subset \mathcal{O}$ such that $g = \delta f$.

Causal Poincaré Lemma

Let $g \in \mathcal{D}_1(\mathbb{R}^4)$ be co-closed, $\delta g = 0$, and $\operatorname{supp} g \perp \mathcal{O}$ (double cone). There is $f \in \mathcal{D}_2(\mathbb{R}^4)$ with $\operatorname{supp} f \perp \mathcal{O}$ such that $g = \delta f$.

Notation: $\mathcal{C}_1(\mathbb{R}^4) \subset \mathcal{D}_1(\mathbb{R}^4)$ subspace of co-closed elements.

Definition: For $g \in \mathcal{C}_1(\mathbb{R}^4)$, put $A(g) \doteq F(f)$, where $\delta f = g$.

Consistency: $\delta f = g = \delta f'$ implies $f' - f = \delta h$ and consequently $F(f') = F(f)$.

Reformulation of properties of F in terms of A:

- $A : \mathcal{C}_1(\mathbb{R}^4) \to \mathfrak{P}$
- $A(\delta(\delta h)) = 0, \ h \in \mathcal{D}_3(\mathbb{R}^4)$ (homogeneous Maxwell equ.) ✓
- $j(g) \doteq A(\delta dg), \ g \in \mathcal{D}_1(\mathbb{R}^4)$ (inhomogeneous Maxwell equ.)
- $A(g) \mapsto A(g_P), \ P \in \mathcal{P}_+^\uparrow$ for $g \in \mathcal{C}_1(\mathbb{R}^4)$ (covariance)
- $[A(g_1), A(g_2)] = ?$ if $\operatorname{supp} g_1 \perp \operatorname{supp} g_2$ (locality?)

Theorem

Let $g_1, g_2 \in \mathcal{C}_1(\mathbb{R}^4)$ such that $\operatorname{supp} g_1 \perp \operatorname{supp} g_2$. Then
(i) $[A(g_1), A(g_2)] \in \mathfrak{P} \bigcap \mathfrak{P}'$ (center)
(ii) $[A(g_1), A(g_2)] = 0$ if $\operatorname{supp} g_1 \times \operatorname{supp} g_2$

The universal algebra

Heuristic idea: Proceed to abstract unitary operators $V(a,g) \mathrel{\widehat{=}} e^{iaA(g)}$

Definition:

$\mathfrak{G}_0$: unitary group generated by $\{V(a,g) : a \in \mathbb{R}, g \in \mathcal{C}_1(\mathbb{R}^4)\}$, relations

$$V(a_1,g)V(a_2,g) = V(a_1+a_2,g)\,, \quad V(a,g)^* = V(-a,g)\,, \quad V(0,g) = 1$$

$$V(a_1,g_1)V(a_2,g_2) = V(1, a_1 g_1 + a_2 g_2) \ \text{ if } \ \operatorname{supp} g_1 \times \operatorname{supp} g_2$$

$$\lfloor V(a,g), \lfloor V(a_1,g_1), V(a_2,g_2) \rfloor \rfloor = 0, \ \text{for any } g \text{ if } \ \operatorname{supp} g_1 \perp \operatorname{supp} g_2$$

$\mathfrak{V}_0$: complex linear span of the elements of $\mathfrak{G}_0$ (*-algebra)

Note: Let ω be any state on $\mathfrak{V}_0$ with GNS-representation $(\pi, \mathcal{H}, \Omega)$. Then $A \mapsto \|\pi(A)\|_{\mathcal{H}}$ defines a C*-semi-norm on $\mathfrak{V}_0$. If ω is faithful, it is a C*-norm.

Lemma

Let ω be the functional on $\mathfrak{G}_0$ given by $\omega(V) = 0$ for $V \in \mathfrak{G}_0 \backslash \{1\}$ and $\omega(1) = 1$. The canonical extension of this functional to the complex linear span of $\mathfrak{G}_0$ is a faithful state on $\mathfrak{V}_0$.

Sketch of proof: $\omega(c_0 1 + \sum_n c_n V_n) = c_0$ for $V_n \in \mathfrak{G}_0 \backslash \{1\}$ (basis of $\mathfrak{V}_0$ in $\mathfrak{G}_0$); $\omega((c_0 1 + \sum_n c_n V_n)^* (c_0 1 + \sum_{n'} c_{n'} V_{n'})) = |c_0|^2 + \sum_n |c_n|^2 \geq 0.$ QED

Corollary

Let $(\pi, \mathcal{H}, \Omega)$ be the GNS-representation induced by ω. The map $A \mapsto \|\pi(A)\|_{\mathcal{H}}$, $A \in \mathfrak{V}_0$, defines a C-norm on $\mathfrak{V}_0$.*

Definition: The completion of $\mathfrak{V}_0$ with regard to the norm

$$\|A\| \doteq \sup_{\pi, \mathcal{H}} \|\pi(A)\|_{\mathcal{H}} , \quad A \in \mathfrak{V}_0$$

is the universal C*-algebra $\mathfrak{V}$ of the electromagnetic field.

The universal algebra

Does $\mathfrak{V}$ satisfy all Haag-Kastler axioms?

Isotony: $\mathcal{O} \mapsto \mathfrak{V}(\mathcal{O}) \doteq C^*\{V(a,g) : a \in \mathbb{R},\ g \in \mathcal{C}_1(\mathcal{O})\}$; the algebra $\mathfrak{V}$ is, by construction, the C*-inductive limit of these local algebras ✓

Covariance: The invertible maps $\alpha_{\cdot}$ defined by

$$\alpha_P(V(a,g)) \doteq V(a, g_P)\,, \quad P \in \mathcal{P}_+^\uparrow,$$

extend to automorphisms of $\mathfrak{G}_0$, theron to its span $\mathfrak{V}_0$, and then by continuity to $\mathfrak{V}$. Moreover, $\alpha_P(\mathfrak{V}(\mathcal{O})) = \mathfrak{V}(P\mathcal{O})$ by construction. ✓

Locality: By definition $[V(a_1,g_1), V(a_2,g_2)] = 0$ if $\operatorname{supp} g_1 \times \operatorname{supp} g_2$. Hence $[\mathfrak{V}(\mathcal{O}_1), \mathfrak{V}(\mathcal{O}_2)] = 0$ if $\mathcal{O}_1 \perp \mathcal{O}_2$. ✓

Primitivity: Not satisfied since $\mathfrak{V}$ has a non-trivial center. ↯

Vital (difficult) step: Pick a suitable pure state $\omega \in \mathfrak{V}^*$ (*e.g.* vacuum) such that the kernel $\ker \pi$ of its GNS-representation is Poincaré invariant. The quotient algebra $\mathfrak{V}/\ker \pi$ then satisfies all Haag-Kastler axioms, *i.e.* defines a theory.

Representations

Characterization of states of interest:

Definition: Let ω be a state on $\mathfrak{V}$.

- ω is regular (strongly regular) if all functions

$$a_1, \ldots, a_n \mapsto \omega(V(a_1, g_1) \cdots V(a_n, g_n))$$

  are continuous (smooth, with tempered derivatives at 0)
- ω satisfies condition L if it is strongly regular and

$$\frac{d}{da} \omega(V_1 V(a, g_1) V(a, g_2) V(a, -g_1 - g_2) V_2)\Big|_{a=0} = 0$$

Theorem

Let ω be a state on $\mathfrak{V}$, satisfying L, with GNS representation $(\pi, \mathcal{H}, \Omega)$

- *There exist selfadjoint operators $A_\pi(g)$ with common stable core $\mathcal{D} \subset \mathcal{H}$ such that $\pi(V(a, g)) = e^{iaA_\pi(g)}$, $a \in \mathbb{R}$, $g \in \mathcal{C}_1(\mathbb{R}^4)$.*
- $a_1 A_\pi(g_1) + a_2 A_\pi(g_2) = A_\pi(a_1 g_1 + a_2 g_2)$ *on* $\mathcal{D}$.

Definition: A state ω on $\mathfrak{V}$ describes the vacuum if it is pure and

- $\omega \circ \alpha_P = \omega, \;\; P \in \mathcal{P}_+^\uparrow$
- $P \mapsto \omega(V_1\, \alpha_P(V_2))$ continuous, $V_1, V_2 \in \mathfrak{V}$
- supp $\{k \mapsto \int dx\, e^{-ikx} \omega(V_1\, \alpha_x(V_2))\} \subset \overline{V}_+, \;\; V_1, V_2 \in \mathfrak{V}$

Theorem

Let ω be a vacuum state on $\mathfrak{V}$ with GNS representation $(\pi, \mathcal{H}, \Omega)$. There exists a continuous unitary representation U_π of $\mathcal{P}_+^\uparrow$ such that

$$U_\pi(P)\pi(V)U_\pi(P)^{-1} = \pi \circ \alpha_P(V)\,, \quad P \in \mathcal{P}_+^\uparrow,\; V \in \mathfrak{V}\,.$$

Note: π is irreducible and $\ker \pi$ is Poincaré invariant; hence $\mathfrak{V}/\ker\pi$ defines a Haag-Kastler theory with dynamics induced by U_π.

Remarks:

- Any vacuum state ω is fixed by its generating functional

$$g \mapsto \omega(V(1,g))\,, \quad g \in \mathcal{C}_1(\mathbb{R}^4)\,.$$

Sketch of proof: Given $g_1, \ldots, g_n$ there are $x_1, \ldots, x_n \in \mathbb{R}^4$ such that

$$\omega(\alpha_{x_1}(V(a_1,g_1)) \cdots \alpha_{x_n}(V(a_n,g_n))) = \omega(V(1, \textstyle\sum_m a_m\, g_{m\, x_m})).$$

Right hand side fixed by generating functional, left hand side can be continued to $x_1 = \cdots = x_n = 0$ by the EOW-Theorem.

- Any vacuum state ω satisfying condition L fixes correlation functions of F satisfying all Wightman axioms.

Examples

Task: Determination of states of interest in $\mathfrak{V}^*$ with property L

(1) *Zero current:* (recall $A_\pi(\delta dg) = j_\pi(g), \; g \in \mathcal{D}(\mathbb{R}^4)$)

Lemma

Let ω_0 be a vacuum state on $\mathfrak{V}$ with zero current. Then

$$g \mapsto \omega_0(V(1,g)) = e^{-c\,\langle g,g\rangle}\,, \quad g \in \mathcal{C}_1(\mathbb{R}^4),$$

(free electromagnetic field in Fock space representation, $c \geq 0$).

(2) *Classical (central) currents:*

Lemma

Let ω be a pure state on $\mathfrak{V}$ in presence of a classical current. Then

$$g \mapsto \omega(V(1,g)) = e^{i\,j_\pi(G_0 g)}\,\omega_0(V(1,g))\,, \quad g \in \mathcal{C}_1(\mathbb{R}^4),$$

(j_π distribution, G_0 Green's function of $\Box$, ω_0 state with zero current)

(3) *Quantum currents:* No rigorous examples yet of vacuum states on $\mathfrak{V}$.

- Feynman (path integral) approach relies on heuristic formula
$$g \mapsto \omega(V_T(1,g)) \doteq Z^{-1} \int dA d\psi d\bar{\psi}\, e^{iS(A,\psi,\bar{\psi})}\, e^{iA(g)}$$
for time ordered exponentials
- Steinmann approach for expectations of unordered exponentials
$$g \mapsto \omega(V(1,g))$$
relies on field equations

Both approaches consistent only in renormalized perturbation theory.

(4) *Topological charges:*

There exist pure states on $\mathfrak{V}$ such that $\pi(\lfloor V(1,g_1), V(1,g_2) \rfloor) \neq 1_{\mathcal{H}}$ for certain $g_1, g_2 \in \mathcal{C}_1(\mathbb{R}^4)$ with supp $g_1 \perp$ supp g_2.

Summary

- Universal C*-algebra $\mathfrak{V}$ of the electromagnetic field has been constructed
- Haag-Kastler axioms satisfied (primitivity)
- Any relativistic QFT involving electromagnetic field induces a particular vacuum state on $\mathfrak{V}$ ($\mathfrak{V}/\ker\pi$ satisfies all axioms)
- Algebra $\mathfrak{V}$ meaningful starting point for study of existence problems and structural analysis (IR problems *etc*)
- Topological features of intrinsic vector potential *A* encoded in center of $\mathfrak{V}$
- AQFT is capable of describing concrete physical systems; approach complementary to LQFT (π versus $\mathcal{L}$)